Prof. Dr. Delio Mugnolo, Dr. Hafida Laasri, Dr. Joachim Kerner

Modul 61213

Funktionalanalysis

LESEPROBE

Fakultät für
Mathematik und Informatik

FernUniversität in Hagen

Liebe Studentin, lieber Student,

wir freuen uns über Ihr Interesse für den Kurs Funktionalanalysis und wünschen Ihnen Erfolg bei der Bearbeitung und Vergnügen beim Lernen.

Funktionalanalysis – was soll das sein? Schließlich haben Sie das gesamte bisherige Studium hindurch reichlich viele Funktionen getroffen, ohne dass man auf die Idee kommen würde, ihnen einen ganzen Kurs zu widmen. Nein, die Funktionalanalysis hat wenig mit allgemeinen Funktion*en* zu tun, vielmehr ist sie die Lehre von den Funktion*ale* – beziehungsweise ist sie als solche entstanden. Funktionale sind wiederum auch keine allzu geheimnisvollen Objekte: Sie sind nichts anderes als Abbildungen von $\mathbb{R}^n$ nach $\mathbb{R}$, beziehungsweise ihre natürlichen Verallgemeinerungen auf unendlichdimensionale Vektorräume. Fordert man darüber hinaus, dass sie eine geeignete Linearitätsbedingung erfüllen, so werden Funktionale zu hoch interessanten mathematischen Objekten mit erstaunlich vielen Anwendungen sowie bemerkenswerten subtilen Problematiken – lineare Funktionale auf unendlichdimensionalen Vektorräumen müssen zum Beispiel nicht stetig sein, so lang man bereit ist, das gängige axiomatische System um das Auswahlaxiom zu ergänzen!

Erstmals untersucht wurden Funktionale zu Beginn des 20. Jahrhunderts. Die Welt der Mathematik war damals ein interessanter, widerspruchsreicher Ort: Viele Forscher (und leider nur sehr wenige Forscherinnen) kämpften um die korrekte Gestaltung der Fundamente obwohl – oder gerade *weil*? – sie gleichzeitig immer wieder auch viele Probleme instinktiv lösten, die erst viel später akkurat untersucht, ja sogar genau formuliert werden konnten. Dies springt besonders ins Auge wenn man bedenkt, dass viele Gruppen – etwa die Göttinger Schule um David Hilbert – tiefe Resultate in der Analyse von Integralgleichungen erzielten, ohne jedoch über den Begriff von unendlichdimensionalem Vektorraum oder von Norm zu verfügen.

Funktionalanalysis also als Fokussierung der klassischen Analysis auf lineare Abbildungen. Gleichzeitig lässt sich die Funktionalanalysis aber auch als Disziplin beschreiben, in der die analytischen Eigenschaften der klassischen Objekte der linearen Algebra untersucht werden. Was nämlich die Analysis von der linearen Algebra im Kern unterscheidet, ist die Möglichkeit – die in allgemeinen Vektorräumen nicht gegeben ist –, Grenzwerte zu bilden und Stetigkeit zu definieren. Somit bilden normierte Räume – das Thema unserer Kurseinheit 2 – gewissermaßen die denkbar elementarsten Objekte, die aus einer Hybridisierung von Analysis und linearer Algebra entstehen können. Normierte Räume wurden somit schnell zu einem der beliebtesten Spielplätze der klassischen Funktionalanalysis und sind in diesem Kurs allgegenwärtig.

Die stürmischen Entwicklungen der Jahre 1895-1915 zeigten deutlich, wie wichtig und vielversprechend ein abstrakterer, ja von der linearen Algebra inspirierter Zugang zur Analysis sein konnte. 100 Jahre alte Probleme der Fourier-Analyse konnten in einem neuen algebraischen Rahmen aufgefasst und gelöst, die Struktur zahlreicher Gegenbeispiele erläutert, die Approximierbarkeit stetiger Funktionen durch Polynome oder trigonometrische Reihen strukturell untersucht werden. In diesen Jahren wurden vor allem in Deutschland und Frankreich, aber auch in Österreich-Ungarn und Italien, etwa Funktionen- und Folgenräume, Lebesguesche Integrationstheorie, Konvergenz in allgemeinen topologischen Räumen eingeführt. Sie blieben jedoch für eine Weile weitgehend getrennte Gebiete. Die Strukturierung und die Systematisierung dieser und weiterer Theorien ist vor allem einer Gruppe polnischer Mathematiker um Stefan Banach und Hugo Steinhaus in Lwów (bis 1918: Lemberg, Österreich-Ungarn; heute Lwiw, Ukraine) zu verdanken, von denen bis 1945 viele starben oder emigrierten.

Die goldenen Jahre der klassischen Funktionalanalysis umfassen die Zeit zwischen der Promotion von Stefan Banach im Jahr 1922 und dem Einmarsch der Wehrmacht in Lwów im Jahr 1941. Somit ist sie auch, stärker und offensichtlicher als wohl die meisten anderen mathematischen Bereiche, ein Spiegel der europäischen Geschichte im 20. Jahrhundert; kaum ein anderes Gebiet wurde so stark von Diktaturen, Kriegen, antisemitischen Verfolgungen, Verschiebungen von Grenzen und Flucht beeinflusst; eine sehr gute und informierte Quelle über die Wechselwirkung von Politik und Funktionalanalysis ist [**?**].

Die Anfänge der Funktionalanalysis wurden von Frankreich und Deutschland besonders geprägt, den beiden Staaten, die an an der Jahrhundertwende de facto das Monopol der Analysis weltweit innehatten; die ersten Versuche im Bereich der Funktionalanalysis waren vor allem darauf gerichtet, alte analytische Probleme mit neuen Methoden zu behandeln. In der Zeit zwischen den Weltkriegen entwickelte sich die Funktionalanalysis vor allem unter dem Impuls der Mitglieder der bereits erwähnten Lemberger Mathematikerschule, deren Interessen eher abstrakt waren und sich von der Wechselwirkung von Topologie und linearer Algebra ableiteten. Nach dem zweitem Weltkrieg nahmen auch die USA und die Sowjetunion, die mittlerweile auch mathematische Mächte geworden waren, selber an der Entwicklung des Gebietes maßgeblich teil, was wieder zu einer Verschiebung der Schwerpunkte führte: Funktionalanalysis wurde nun vor allem als Werkzeug für Numerik, Approximationstheorie, Differenzialgleichungen, Stochastik und vor allem mathematische Physik verstanden. Denn es ist ein großes Wunder der Natur, dass die Funktionalanalysis und ihre Tochterdisziplinen die Sprache der Quantenmechanik sind. Wenig, was nach 1950 in der Funktionalanalysis passiert ist, wird aber in diesem Kurs abgebildet.

Und nun zur Struktur dieses Kurses: Er besteht aus sieben Einheiten über

- Metrische Räume
- Normierte Räume
- Lineare Operatoren
- Funktionale und schwache Konvergenz
- Lebesgue- und Sobolevräume
- Hilberträume
- Spektraltheorie

Wir empfehlen Ihnen, die Kapitel in dieser Reihenfolge durchzuarbeiten, da sie meistens aufeinander aufbauen. Die Kapitel werden getaktet durch das Abwechseln von lockereren Texten mit formalisierten Inhalten. Zum einem haben wir versucht, die historischen Entwicklungen zu schildern, die dazu geführt haben, dass einige Begrifflichkeiten oder sogar ganze Forschungsrichtungen sich herauskristallisiert haben; zum anderen präsentieren wir die wichtigsten Sätzen der Funktionalanalysis zusammen mit ihren Beweisen, welche oft ihre ganz eigenen interessanten Geschichten aufweisen. Darüber hinaus werden Sie im Text zahlreiche Übungsaufgaben finden: Sie selbständig zu lösen wird Ihnen helfen, die Begriffe und die Beweismethoden im Lernstoff besser zu verinnerlichen. (Die zugehörigen Lösungsvorschläge finden Sie dann am Ende der jeweiligen Kapitel.) Ebenfalls am Ende jedes Kapitels finden sie einen kurzen Abschnitt über *Anmerkungen und Empfehlungen fürs weitere Lernen*: Sie sind eine Auswahl jener Themen, die aus Platzmangel nicht in den Haupttext aufgenommen wurden und trotzdem interessant sind – ob wegen der eleganten Beweise und der Verschärfung von Resultaten im Haupttext oder aufgrund der unerwarteten Einblicke in benachbarte Gebiete, die sie erlauben. Sie bieten durchaus mögliche Themen für ein Seminar oder für eine Abschlussarbeit.

Nachdem die wichtigsten Inhalte dieses Kurses zusammengefasst wurden, sollte nicht unerwähnt bleiben, was in diesem Kurs *nicht* behandelt wird, darunter:

- **Unbeschränkte Operatoren:** Im Text werden ausschließlich Resultate über beschränkte lineare Operatoren präsentiert. Dabei haben unbeschränkte Operatoren seit den 1930er Jahren sehr viel Aufmerksamkeit bekommen, auch angesichts ihrer Rolle in der Quantenmechanik. Der Kurs 1347 „Lineare Operatoren im Hilbertraum“ ist vor allem der Spektraltheorie dieser Operatoren so wie ihrer Anwendung gewidmet.

- **Nichtlineare Funktionalanalysis:** Einige der Resultate, die Sie in diesem Skript finden, gelten nicht nur für lineare Operatoren. Die meisten Methoden topologischer Natur können tatsächlich den nichtlinearen Kontexten erweitert werden, außerdem kann man manchmal die Linearität eines Funktionals durch die schwächere Bedingung der Konvexität ersetzen. Bei genügendem Interesse könnte gerne ein Seminar über nichtlineare Funktionalanalysis gerne angeboten werden.

- **Anwendungen in der Theorie der linearen Differenzialgleichungen:** Funktionalanalytische Methoden für die Untersuchung von partiellen Differenzialgleichungen gehören seit Jahrzehnten zur Grundausstattung der angewandten Mathematik. Außerdem sind sie interessant, weil sie eine Wechselwirkung von Hilbertraummethoden, Sobolevräumen und Ad-hoc-Argumenten, die von der relevanten Klasse von Differenzialgleichungen abhängen, erfordern. Viele dieser Themen werden in den kommenden Jahren im Skript zum Kurs 1380 „Partielle Differenzialgleichungen“ Eingang finden.

Eine letzte, wichtige Anmerkung zum Schluss: Wie Sie schon im Modulhandbuch gelesen haben, ist dieses Skript in englischer Sprache verfasst. Es gibt verschiedene Gründe für unsere Entscheidung, einen Kurs an einer deutschen Universität von einem Lehrtext auf Englisch begleiten zu lassen: Zum einen gehört seit vielen Jahrzehnten die Fähigkeit, mathematische Texte in englischer Sprache zu verstehen, zu den Kernexpertisen jeder Mathematikerin und jedes Mathematikers; zum anderen wird die Erfahrung mit mathematischem Englisch ihre erste Phase bei einer Abschlussarbeit deutlich vereinfachen, da seit längerer Zeit praktisch alle Forschungsartikel, die Sie dafür lesen können müssen, eben in englischer Sprache verfasst sind. Sie sollten auch nicht vergessen, dass bereits jetzt die meisten Seminarreferate auf Fachliteratur in englischer Sprache beruhen.
Wir haben versucht, Ihren Einstieg ins mathematische Englisch möglichst zu vereinfachen: Am Anfang jedes Kapitels finden sie eine kurze Zusammenfassung der relevanten Resultate und Lernziele auf Deutsch; darüber hinaus wird allen (englischen) Einträgen im Glossar eine deutsche Übersetzung beigefügt. Selbstverständlich werden Klausur und mündliche Prüfungen zu diesem Kurs auf Deutsch formuliert beziehungsweise abgenommen.

In diesem Sinne: *have fun!*

DR. JOACHIM KERNER

DR. HAFIDA LAASRI

PROF. DR. DELIO MUGNOLO

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Studierhinweise zu Kurseinheit 1

In dieser Kurseinheit befassen wir uns mit einem zentralen Aspekt der Funktionalanalysis, den metrischen Räumen. Das sind Mengen, auf denen ein gewissen Axiomen genügender Abstandsbegriff gegeben ist. Wir werden topologische Grundbegriffe wie den der *offenen Menge, abgeschlossenen Menge, Umgebung eines Punktes* einführen, die dann für die weiteren Kurseinheiten grundlegend sind. Einige Definitionen und Begriffe sind Ihnen bereits aus der reellen Analysis bekannt. In dieser Kurseinheit werden diese aber entsprechend verallgemeinert. Ferner werden *vollständige* und *kompakte* metrische Räumen ausführlich diskutiert. Wir setzen etwas Vertrautheit mit den Begriffen *Grenzwert einer Folge*, *Stetigkeit einer Funktion* sowie *Isomorphismus, Kontraktion* u.a. voraus. Besonders zentrale Aussagen dieser Kurseinheit sind der *Satz von Bolzano-Weierstrass* und der *Banachsche Fixpunktsatz.*

Lernziele:

Abschnitt 1.1: Definition und Beispiele (Definition and examples)

Nach der Lektüre dieses Paragraphen sollten Sie in der Lage sein, den Begriff der Metrik und des metrischen Raumes definieren und an Beispielen erläutern zu können. Wichtig sind vor allem Beispiele, die über den $\mathbb{R}^n$ hinausgehen.

Abschnitte 1.2: Topologische Begriffe (Topological concepts)

In diesem Abschnitt wird die topologische Struktur metrischer Räume diskutiert. Nach der Bearbeitung dieses Paragraphen sollten Sie

- die Begriffe: offene (abgeschlossener) Kugel (Menge), Umgebung, Abschluss einer Menge verstanden haben und erklären können.
- wissen, dass eine Metrik eine topologische Struktur induziert.
- wissen, wie das System der offenen Mengen mit dem System der abgeschlossenen Mengen bzw. dem des Umgebungssystems zusammenhängt.

Abschnitte 1.3: Folgen in metrischen Rämen (Sequences in metric spaces)

In diesem Abschnitt werden wir lernen, dass eine Metrik nicht nur eine topologische Struktur, sondern auch eine *uniforme Struktur* induziert. Nach Durcharbeiten dieses Abschnitts sollten Sie

- mit den Begriffen: konvergente Folge, Teilfolge, Cauchy-Folge, vollständiger metrischer Raum umgehen und an Beispielen erläutern können.

- wissen, wie abgeschlossene Mengen mittels der Konvergenz von Folgen charakterisiert werden.
- die Beziehung zwischen konvergente Folgen, konvergente Teilfolgen und Cauchy-Folgen aufzeigen können;
- in der Lage sein, ein Beispiel einer Cauchy-Folge, die nicht konvergiert, angeben zu können.
- wissen, wann ein Teilraum eines vollständigen metrischen Raumes selbst vollständig ist.
- in der Lage sein nachzuweisen, dass verschiedene Metriken auf einer Menge zwar dieselbe Topologie, aber unterschiedliche uniforme Strukturen erzeugen können.

Abschnitte 1.4: Stetige Abbildungen (Continuous functions)

In diesem Abschnitt werden wir uns mit verschiedenen Begiffen der Stetigkeit einer Abbildung befassen. Nach dem Durcharbeiten dieses Abschnitts sollten Sie

- in der Lage sein, die Definition der Stetigkeit, der gleichmäßigen Stetigkeit und der Lipschitz-Stetigkeit einer Abbildung anzugeben. Desweiteren sollten Sie Kriterien für Stetigkeit angeben und diese anwenden können.
- wissen, dass Stetigkeit und Folgenstetigkeit einer Abbildung in metrischen Räumen äquivalente Konzepte sind.

Abschnitt 1.5: Fortsetzung von stetigen Funktionen (Extension by density of continuous functions)

Nach dem Durcharbeiten dieses Abschnittes sollte Ihnen klar geworden sein, dass man sich gleichmässig stetige Abbildungen, die auf einem dichten Teilraum definiert sind, eindeutig auf den ganzen Raum fortsetzen lassen.

Abschnitt 1.6: Separable metrische Räume (Separable metric spaces)

Nach dem Durcharbeiten dieses Abschnitts sollten Sie wissen, was man unter einem separablen metrischen Raum versteht und Kriterien für Separabilität formulieren können.

Abschnitt 1.7: Kontraktionsprinzip (Banach fixed-point theorem)

In diesem Abschnitt werden Existenz- und Eindeutigkeitssätze von Fixpunkten einer Abbildung behandelt. Nach der Lektüre dieses Abschnitts sollten Sie in der Lage sein, den Banachschen Fixpunktsätz formulieren, beweisen und anwenden zu können.

Abschnitt 1.8: Kompakte metrische Räume (Compact metric spaces)

Ein weiteres zentrales Konzept der Funktionalanalysis, welches wir in diesem letzten Abschnitt betrachten, ist das der Kompaktheit. Nachdem Sie diesen Abschnitt durchgearbeitet haben, sollten Sie

- wissen, was man unter einem kompakten Raum und was unter einer kompakten bzw. relativ-kompakten Teilmenge versteht.
- die Eigenschaften kompakter Räume (Teilmengen) aufzählen und diese nachweisen können.

Darüber hinaus sollten Sie nachweisen können,

- dass in einem metrischen Raum kompakte Mengen abgeschlossen und beschränkt sind und die Umkehrung (im Gegensatz zum $\mathbb{R}^n$) im Allgemeinen nicht gilt.
- dass stetige Funktionen auf kompakten Mengen gleichmäßig stetig sind und dass, im reellwertigegen Fall, das Supremum sowie das Infimum angenommen werden.

Chapter 1

Metric spaces

In this chapter we will introduce the important concept of metric spaces and give several examples thereof. If functional analysis is a hybrid of linear algebra and analysis, then metric spaces may look out of place: no sums or further operations are in general defined, only distances can be measured. This is enough to discuss limiting processes and more general topological issues, but not to allow for composition of vectors or to introduce the notion of linearity. Nevertheless, we will present in this chapter a manifold of objects and also prove several results that will play an important role in all later chapters. This introduction will also allow us to follow the earliest historical developments of functional analysis and to meet some of its pioneers, like Maurice Fréchet and Stefan Banach.

Introduced in 1906 in Paris by Maurice René Fréchet, metric spaces are sets on which a (meaningful) distance can be defined. The most well-known example of a metric space is the Euclidean space $\mathbb{R}^n$ where by the Pythagorean theorem the distance $d_2(x, y)$ of two points $x, y \in \mathbb{R}^n$ can be calculated via

$$d_2(x,y) = \Big(\sum_{j=1}^{n} |x_j - y_j|^2\Big)^{\frac{1}{2}} .$$

This particular distance is meaningful since it is positive and equals zero only if and only if the two points x and y are identical. Furthermore, it fulfills some minimizing property in the sense that $d_2(x, y)$ is the least distance one has to "walk" to get from x to y. In other words, if $z \in \mathbb{R}^n$ is any additional point one has

$$d_2(x,y) \le d_2(x,z) + d_2(z,y) .$$

Also, as you have already learned in earlier courses, concepts like convergence and continuity of functions $f : \mathbb{R}^n \to \mathbb{R}$ can be defined in terms of the *metric* d_2. For example, a function $f : \mathbb{R} \to \mathbb{R}$ is continuous at $x_0 \in \mathbb{R}$ if for any $\varepsilon > 0$ there exists a $\delta > 0$ such that, $d_2(x, x_0) < \delta$ implies

$$d_2(f(x), f(x_0)) < \varepsilon \qquad \text{for all } x \in X.$$

The crucial insight in defining a distance on more general sets is then to realize that the explicit form of the distance function d is not that important. As we will see, only a few requirements are necessary to determine a "good" metric d which then allows to transfer important concepts such as the two mentioned above from Euclidean to more general spaces. In addition to that, we know that the topology of Euclidean space is also determined by the metric d_2. Namely, a subset $A \subset \mathbb{R}^n$ is called *open* whenever for each point $x \in A$ there exists $r > 0$ such that

$$B_{\mathbb{R}^n}(x,r) := \{y \in \mathbb{R}^n : d_2(x,y) < r\}$$

is contained in A. However, this construction can be made for all metric spaces and we hence see that a metric d on *any* set X also defines a topology in a very natural way.

1.1 Definition

In the following $\mathbb{K}$ will denote either the real field $\mathbb{R}$ or the complex field $\mathbb{C}$ and X shall always be a non-empty set.

Definition 1.1.1. *A* ***metric*** *or* ***distance*** d *on* X *is a mapping*

$$\begin{aligned} d:\ X \times X &\to \mathbb{R} \\ (x,y) &\mapsto d(x,y) \end{aligned}$$

such that for all x, y, z *in* X, *the following conditions are satisfied:*

(M1) $d(x,y) = 0$ *if and only if* $x = y$,

(M2) $d(x,y) = d(y,x)$,

(M3) $d(x,z) \leq d(x,y) + d(y,z)$.

A pair (X,d) *of a set* X *and a metric* d *on* X *is called a* ***metric space***. *We call* $d(x,y)$ *the distance between* x *and* y *(with respect to the metric* d*).*

One usually refers to properties (M1), (M2), (M3) by saying that a distance mapping is *positive definite*, *symmetric* and satisfies the *triangle inequality*, respectively.

Remark 1.1.2. *If* (X,d) *is a metric space, then* d *is necessarily a non-negative mapping, i.e.,* $d(x,y) \geq 0$ *for all* $x, y \in X$. *This is an easy consequence of the axioms (M1), (M2), (M3). In fact, for all* $x, y \in X$ *we have*

$$0 = d(x,x) \leq d(x,y) + d(y,x) = 2d(x,y).$$

Furthermore, the inequalities

$$|d(x,z) - d(y,z)| \leq d(x,y) \qquad \text{for all } x, y, z \in X\ , \tag{1.1}$$

and

$$|d(x,y) - d(u,v)| \leq d(x,u) + d(y,v) \qquad \text{for all } x, y, u, v \in X \tag{1.2}$$

hold.

Let us emphasize that different metrics on the same set X can exist, cf. Example 1.1.3. However, sometimes there is no confusion about what metric is considered and one simply writes X referring to a metric space without specifying d (the most prominent example here is Euclidean space $\mathbb{R}^n$). We now want to give several standard examples of metric spaces some of which will appear repeatedly in later chapters.

Example 1.1.3. *1) The **Euclidean metric** $d_2 : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ on $\mathbb{R}^n$ is given by*

$$d_2(x, y) := \left(\sum_{i=1}^{n} |x_i - y_i|^2 \right)^{1/2}$$

with $x = (x_1, x_2, ..., x_n)$ and $y = (y_1, y_2, ..., y_n)$ in $\mathbb{R}^n$. For $n = 1$, the Euclidean metric reduces to the usual absolute-value distance $d(x, y) = |x - y|$ on the real line.

Also the mappings

$$d_1(x, y) := \sum_{i=1}^{n} |x_i - y_i| \quad \text{and} \quad d_\infty(x, y) := \max_{1 \le i \le n} |x_i - y_i|$$

*define metrics on $\mathbb{R}^n$. They are called the **Manhattan metric** and the **maximum metric** on $\mathbb{R}^n$, respectively. Can you imagine why d_1 is called so? If not, try to compute the distance between, say, $(0, 0)$ and $(12, 3)$ with respect to d_1, and then take a look at a map of Manhattan (or downtown Mannheim, for that matters) and then think of how distant two junctions are. More generally,*

$$d_p(x, y) := \left(\sum_{i=1}^{n} |x_i - y_i|^p \right)^{1/p}$$

defines a metric on $\mathbb{R}^n$ for all $p > 0$.

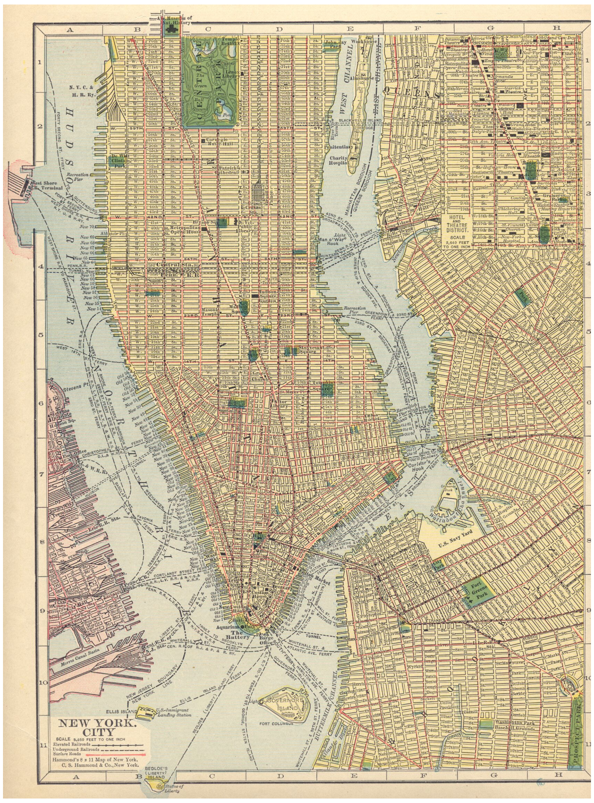

Figure 1.1: A map of Manhattan in 1910

2) On $\mathbb{R}^n$ the mapping

$$\mathbf{d}(x, y) := \begin{cases} d_2(x, y) & \text{if } x = \kappa y \text{ for some } \kappa \in \mathbb{R} \\ d_2(x, 0) + d_2(y, 0) & \text{otherwise} \end{cases} \tag{1.3}$$

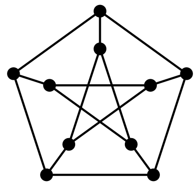

Figure 1.2: A finite graph with 10 vertices and 15 edges

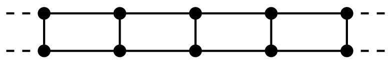

Figure 1.3: An infinite graph

defines a metric on $\mathbb{R}^n$ *where* d_2 *is the Euclidean metric. The metric* d *is called* ***SNCF metric*** *because in the French railway system (managed by France's national railway company SNCF), the fastest connection between two cities* x *and* y *(e.g., from Bordeaux to Lyon) usually goes through the origin (Paris of course) unless* $y = \kappa x$*, i.e., they already lie on the same TGV line (e.g., Lyon and Marseille).*

3) *A rather trivial example of a metric on an arbitrary non-empty set* X *is given by the* ***discrete metric*** *defined by*

$$d(x,y) := \begin{cases} 1 & \textit{if } x \neq y \\ 0 & \textit{if } x = y \end{cases} \qquad x, y \in X \ . \tag{1.4}$$

In particular, any non-empty set X *is* ***metrizable****, i.e, there exists a mapping* $d : X \times X \to \mathbb{R}$ *that turns* X *into a metric space* (X, d)*.*

4) *Let us consider the set* $X := \mathbb{Z}$ *of integer numbers and the mapping* $d_{\mathbb{Z}}$ *defined by* $d_{\mathbb{Z}}(n,m) := |n - m|$ *for all* $n, m \in \mathbb{Z}$*. Then* $(\mathbb{Z}, d_{\mathbb{Z}})$ *is a metric space.*

5) *A (simple, undirected)* ***graph*** *is a pair* $G = (V, E)$ *composed of a finite or countable set* V*, whose elements are called* ***vertices****, and another set* $E \subset \{\{v, w\} : v, w \in V, v \neq w\}$*, whose elements are called* ***edges****): if* $e = \{v, w\}$ *is an element of* E*, then the vertices* v, w *are said to be* ***adjacent*** *(by means of* e*).*

   A ***path*** *of* ***length*** m *from* v *to* w *is a sequence*

$$v_0, e_1, v_1, e_2, \cdots, e_m, v_m$$

   of vertices $v := v_0, v_1, \cdots, v_m =: w \in V$ *and edges* $e_1, e_2, \cdots, e_m \in E$ *such that* v_i *and* v_{i+1} *are adjacent by means of* e_{i+1}*. The graph is called* ***connected*** *if any two vertices are joined by at least one path.*

   If G *is connected, then the function* $d_G : V \times V \to \mathbb{N}$ *defined by*

$$d_G(v, w) := \inf\{\textit{length of } P : P \textit{ is a path from } v \textit{ to } w\}$$

   turns V *into a metric space* (V, d_G)*;* d_G *is called the* ***shortest-path metric****.*

   We emphasize that this construction strictly generalizes two already considered notions. Indeed, if (X, d) *is a discrete metric space, i.e.,* d *is the discrete metric, then we can turn it into a graph with vertex set* $V := X$ *and such that any two vertices are adjacent, so that* $d_G(v, w) = 1$ *if and only if* $v \neq w$*. Then* $d_G = d$ *and thus* (X, d) *coincides with* (V, d_G)*.*

Furthermore, we can look at $\mathbb{Z}$ as a graph, letting $V := \mathbb{Z}$ and regarding two numbers/vertices n, m as adjacent if $|n - m| = 1$, i.e., if n, m are subsequent numbers. Then, the metric spaces $(\mathbb{Z}, d_{\mathbb{Z}})$ and (V, d_G) agree.

6) *It is known that a matrix $M \in \mathcal{M}_{n,m}(\mathbb{R})$ has rank 0 if and only if $M = 0$. Using this, one sees that the mapping*

$$d(M, N) := \textit{rank}\ (M - N), \qquad M, N \in \mathcal{M}_{n,m}(\mathbb{R})\ ,$$

defines a metric on $\mathcal{M}_{n,m}(\mathbb{R})$.

7) *Let $C(K, \mathbb{K})$, or simply $C(K)$, denote the set of continuous function $f : K \to \mathbb{K}$ where $K \subset \mathbb{R}$ is compact. For example we could take K to be the closed bounded interval $[a, b]$. Then both*

$$(f, g) \mapsto \int_K |\ f(x) - g(x)\ |\,\mathrm{d}x \quad \textit{and} \quad (f, g) \mapsto \sup_{x \in K} |f(x) - g(x)|.$$

are metrics on $C(K)$.

8) *Let (X, d) be a metric space. Then*

$$d_1(x, y) := \frac{d(x, y)}{1 + d(x, y)} \quad \textit{and} \quad d_2(x, y) := \min\{d(x, y), 1\}$$

are also metrics on X. More generally, if d is a metric then $\varphi \circ d$ is also a metric provided that $\varphi : \mathbb{R}_+ \to \mathbb{R}_+$ is a non-decreasing mapping such that

$$\varphi(s + r) \le \varphi(s) + \varphi(r),$$

$\varphi(0) = 0$ and $\varphi(s) > 0$ for $s > 0$.

9) *Let (X, d) be a metric space and let A be a subset of X. The restriction d_A of d to $A \times A$, that is*

$$d_A(x, y) := d(x, y), \qquad x, y \in A,$$

*defines a metric on A, called **induced metric**. Whenever a metric space (X, d) is given and we wish to generate a metric space structure on a subset $A \subset X$, we will always canonically endow A with the relative metric d_A, unless otherwise stated.*

1.1.1 Exercises

E 1.1.4. *Show that the inequalities* (1.1) *and* (1.2) *hold.*

E 1.1.5. *Let X be the **extended real line** $\mathbb{R} \cup \{\pm\infty\}$ and set*

$$d_X(x, y) := |\arctan x - \arctan y|, \quad \forall x, y \in X,$$

where

$$\arctan(\pm\infty) := \lim_{x \to \pm\infty} \arctan(x) = \pm\frac{\pi}{2}\ .$$

Show that (X, d_X) is a metric space.

E 1.1.6. *Let*

$$\ell^\infty(\mathbb{N}) := \{(x_n)_{n\in\mathbb{N}} \subset \mathbb{R} : \sup_{n\in\mathbb{N}} |x_n| < \infty\}$$

be the space of bounded real sequences. For

$$d_\infty(x, y) := \sup_{n\in\mathbb{N}} |x_n - y_n|,$$

check if (ℓ^∞, d_∞) *is a metric space.*

E 1.1.7. *Consider the set*

$$c_{00}(\mathbb{N}) := \{(x_n)_{n\in\mathbb{N}} \in \ell^\infty(\mathbb{N}) : \exists n_0 \textit{ such that } x_n = 0\ \forall n \geq n_0\}$$

of all eventually vanishing sequences of real numbers, which is clearly a subspace of $\ell^\infty(\mathbb{N})$*. Prove that*

$$d_H(x, y) := \#\{n \in \mathbb{N} \mid x_n \neq y_n\}$$

defines a metric on $c_{00}(\mathbb{N})$ *but not on* $\ell^\infty(\mathbb{N})$*. (This function* d_H *is referred to as* ***Hamming distance****, after the early computer scientist Richard Wesley Hamming: if* x, y *are two pieces of binary code,* $d_H(x, y)$ *says how close they are by measuring at how many entries they differ, possibly up to equalizing their lengths by extending the shorter one by 0.)*

E 1.1.8. *Check if the following maps define a metric on* $\mathbb{R}$ *:*

$i)$ $d_1(x, y) := |x^2 - y^2|$,

$ii)$ $d_2(x, y) := |x^3 - y^3|$,

$iii)$ $d_3(x, y) := e^{\frac{1}{|x-y|}}$.

E 1.1.9. *Show that the space* $C(K)$ *of continuous functions on a compact interval* K *with* d_∞ *from Example 1.1.3 is, in fact, a metric space. (The same assertion also holds for* $C(K)$ *whenever* K *is merely a compact metric space, a notion that generalizes compact intervals which we will meet later on.)*

1.2 Topological concepts

When you first met continuous functions in the courses 1141 and 1144, they were (more or less explicitly) defined by means of an $\varepsilon - \delta$ criterion, but an equivalent definition of continuity is based on the notion of *open set*:

> A function $f : \mathbb{R}^n \to \mathbb{R}^m$ is continuous if and only if the pre-image $f^{-1}(O)$ of each open set $O \subset \mathbb{R}^m$ is an open subset of $\mathbb{R}^n$.

This has a major advantage: it allows for an abstract introduction of the notion of continuity independently of the specific structure of the Euclidean spaces. This sounds good, since the extension of analytical notions and results to sets as general as possible is among the leading ideas of functional analysis. Because open sets were defined by means of neighborhoods, everything boils down to replacing the "classical" Euclidean distance by more general mappings: this has been done in the previous section. For a metric space, we will see that open sets can be constructed from a particular important subclass of open sets, the so called **open balls**. More precisely, the set

$$B(x, r) = \{y \in X : d(x, y) < r\}$$

is called the **open ball of radius** r **and center** x. For example, the open ball in $\mathbb{R}$ of radius r centered at x is the open interval $(x - r, x + r)$. Note that, in the mathematical literature it is also customary to write $B_r(x)$ instead of $B(x, r)$.

Definition 1.2.1. *A set $U \subset X$ is called* ***neighborhood*** *of a point $x \in X$ if there exists $r > 0$ such that $B(x, r) \subset U$.*

The following theorem shows that each metric space is a **Hausdorff space**, i.e., a space in which any two distinct elements admit a pair of disjoint neighborhoods.

Theorem 1.2.2. *Let (X, d) be a metric space. Then for every pair of distinct points $x, y \in X$, there exist neighborhoods U and V of x and y, respectively, such that $U \cap V = \emptyset$.*

Proof. Let x, y be two distinct points of X and let $r := d(x, y) > 0$. Define $U := B(x, \frac{r}{2})$ and $V := B(y, \frac{r}{2})$. Thus U and V are disjoint: if there existed a $z \in U \cap V$ then, using the triangle inequality,

$$r = d(x, y) \leq d(x, z) + d(z, y) < \frac{r}{2} + \frac{r}{2} = r \ .$$

This is absurd, hence $U \cap V = \emptyset$. $\square$

Definition 1.2.3. *Let (X, d) be a metric space. A set $O \subset X$ is called* ***open*** *if for all $x \in O$ there exists $r > 0$ such that $B(x, r) \subset O$.*

It is always important to specify the relevant metric space! The set $[0, 1)$ is e.g. an open set in the metric space $[0, 1]$ (with the induced metric of $\mathbb{R}$), since $[0, 1) = (-1, 1) \cap [0, 1]$. However, $[0, 1)$ is not open in $\mathbb{R}$.

Remark 1.2.4. *Clearly, in a metric space a set is open if and only if it is a neighborhood of each of its elements. A first example of an open set is given by the open ball $B(x_0, r)$ for some $x_0 \in X$ and $r > 0$. In fact, consider $y_0 \in B(x_0, r)$ and choose $r' > 0$ such that $0 < r' < r - d(x_0, y_0)$. For every $y \in B(y_0, r')$*

$$d(y, x_0) \leq d(y, y_0) + d(y_0, x_) \leq r' + d(y_0, x_0) < r.$$

This implies $y \in B(x_0, r)$ and hence $B(y_0, r') \subset B(x_0, r)$. As a consequence, $B(x_0, r)$ is open.

Theorem 1.2.5. *Let (X, d) be a metric space. Then the following assertions hold.*

1) $\emptyset$ and X are open.

2) If $O_1 \subset X$ and $O_2 \subset X$ are two open sets, then $O_1 \cap O_2$ is open.

3) If $(O_i)_{i \in I}$ is an arbitrary family of open sets of X, then the union $\bigcup_{i \in I} O_i$ is open.

Proof. 1) is clear.
2) Let $x \in O_1 \cap O_2$. Then there exists $r_1 > 0$ and $r_2 > 0$ such that $B(x, r_1) \subset O_1$ and $B(x, r_2) \subset O_2$. For $r := \min\{r_1, r_2\} > 0$ we have $B(x, r) \subset O_1 \cap O_2$. This proves that $O_1 \cap O_2$ is open. The assertion follows from (a) if O_1 and O_2 are disjoint.
3) Let $x \in \bigcup_{i \in I} O_i$. Then there exists $j \in I$ with $x \in O_j$. Since O_j is open there exists $r_j > 0$ such that $B(x, r_j) \subset O_j \subset \bigcup_{i \in I} O_i$. $\square$

Remark 1.2.6. *It follows from assertion 2) in Theorem 1.2.5 that finite intersections of open sets are also open. However, this is not true anymore for an infinite number of intersections. For example, in $\mathbb{R}$ with the absolute-value metric, the intervals $(-\frac{1}{n}, 1 + \frac{1}{n})$ with $n > 0$ are open but the intersection*

$$\bigcap_{n=1}^{\infty} \left(-\frac{1}{n}, 1 + \frac{1}{n} \right) = [0, 1]$$

is not open.

Definition 1.2.7. *A set $F \subset X$ is called **closed** if its complement $F^{\complement} := X \setminus F$ is open.*

The **closed ball** with center x and radius $r > 0$ is the set

$$\overline{B}(x,r) := \{y \in X : d(x,y) \leq r\} .$$

Each closed ball is a closed set in X, since its complement is open. Indeed, let $x_0 \in \overline{B}(x,r)^{\complement}$ and set $r_1 := d(x,x_0) - r > 0$. Then for each $y \in B(x_0,r_1)$ we have

$$d(x,x_0) \leq d(x,y) + d(y,x_0)$$

by the triangle inequality and therefore

$$d(x,y) \geq d(x,x_0) - d(x_0,y) > r_1 + r - r_1 = r .$$

This implies that $B(x_0,r_1) \subset \overline{B}(x,r)^{\complement}$ and we conclude that $\overline{B}(x_0,r)^{\complement}$ is in fact open.

Starting from Definition 1.2.7 we can immediately establish an analog of Theorem 1.2.5 for closed sets.

Theorem 1.2.8. *Let (X,d) be a metric space. Then the following assertions hold.*

1) $\emptyset$ and X are closed sets.

2) If $F_1 \subset X$ and $F_2 \subset X$ are two closed sets, then $F_1 \cup F_2$ is closed.

3) If $(F_i)_{i\in I}$ is an arbitrary family of closed sets of X, then the intersection $\bigcap_{i\in I} F_i$ is closed.

Note that $\emptyset$ and X are example of sets that are both open and closed. Furthermore, if a set is not open, this, in general, does not mean that it is closed. For instance, the half-open interval $[0,1)$ is neither open nor closed in $\mathbb{R}$.

Definition 1.2.9. *Let (X,d) be a metric space and let $A \subset X$.*

*1) The set $\mathring{A} := \{x \in X : \exists r > 0$ such that $B(x,r) \subset A\}$ is called the **interior of** A. The elements of $\mathring{A}$ are called **interior points** of A.*

*2) The set $\overline{A} := \{x \in X : \forall r > 0\ B(x,r) \cap A \neq \emptyset\}$ is called the **closure of** A.*

*3) The set $\partial A := \overline{A} \cap \overline{A^{\complement}}$ is called the **boundary of** A.*

*4) A subset $A \subset X$ is said to be **dense in** X (or simply **dense**, if no confusion is possible) if $\overline{A} = X$.*

Proposition 1.2.10. *Let (X,d) be a metric space X and let $A \subset X$. Then the following assertions hold.*

1) The interior $\mathring{A}$ of a set A is the largest open set which is contained in A.

2) The closure $\overline{A}$ of a set $A \subset X$ is the smallest closed set which contains A.

Proof. 1) Let $O \subset X$ be an open set with $A \subset O$. Following Definition 1.2.3 we can find, for any $x \in O$, a $r > 0$ such that $B(x,r) \subset O$. Hence $x \in \overset{\circ}{A}$ and therefore $O \subset A$.

2) The assertion follows from 1) by passing to the complement. □

Proposition 1.2.11. *Let (X,d) be a metric space and let $A \subset X$. Then the following assertions hold.*

1) $\overline{A}^{\complement} = (A^{\complement})^{\circ}$ *and* $(A^{\circ})^{\complement} = \overline{A^{\complement}}$.

2) $\overline{\overline{A}} = \overline{A}$ *and* $\mathring{\mathring{A}} = \mathring{A}$.

3) A is open if and only if $A = \mathring{A}$.

4) A is closed if and only if $A = \overline{A}$.

5) $\partial A = \partial(A^{\complement})$.

6) Let additionally $A \neq X$. *If A is dense, it is not closed. If A is closed, it is not dense.*

Example 1.2.12. *On a metric space* (X, d) *every singleton* $\{x\}$, $x \in X$, *is closed, see Exercise 1.3.19.*

Example 1.2.13. *A topology is called discrete if every subset is open, or equivalently closed (since its complement is open).*

If a non-empty set X *is endowed with the discrete metric introduced in Example 1.1.3, then every subset of* X *is both open and closed. The same holds in general for any metric space in which neighborhoods of a point* x *cannot become arbitrarily small without reducing to the singleton* $\{x\}$*: this is for instance also the case for the metric space* (V, d_G) *where* V *is the set of vertices of a graph* $G = (V, E)$ *and* d_G *the shortest-path metric (see again Example 1.1.3), and more generally for all metric spaces* (X, d) *whose metric* d *can only attain finitely many values. For this reason, in the following we generally call* ***discrete metric spaces*** *all metric spaces* (X, d) *with the property that the set of all values of* d *is a subset of* $\mathbb{R}_+$ *without cluster points.*

Things are however less trivial if G *is turned into a* ***weighted graph*** $G_\mu = (V, E)$ *by associating a weight* $\mu(e) > 0$ *with each* $e \in E$*: then the length of a path between two nodes is defined as the sum of the weights of the edges that appear in the path and the distance between two nodes is modified accordingly. Hence, if* $\inf_{e \in E} \mu(e) = 0$*, then* (V, d_{G_μ}) *is not a discrete metric space any more.*

Remark 1.2.14. *A closed ball* $\overline{B}(x, r)$ *always contains the closure of the open ball* $B(x, r)$*, but they do not coincide in general. For instance, let* X *be a set which contains more than one point equipped with the discrete metric* d*. For* $x \in X$*,*

$$B(x, 1) = \{x\}, \qquad \overline{B}(x, 1) = X$$

and since the set $\{x\}$ *is closed,* $\{x\} = \overline{\{x\}} \neq \overline{B}(x, 1)$.

Definition 1.2.15. *A subset* A *of a metric space* X *is called* ***bounded*** *if it is non-empty and there exist* $x_0 \in X$ *and* $0 \leq R < \infty$ *such that*

$$d(x_0, y) \leq R \quad \text{for all } y \in A. \tag{1.5}$$

We readily see that $A \subset X$ is bounded if and only if there exists $r > 0$ such that $A \subset B(x_0, r)$. This definition is independent of x_0 since, due to the triangle inequality, one has

$$B(x_0, r) \subset B(x, r') \text{ for all } x \in X$$

where $r' = r + d(x_0, x)$. The **diameter** of a set A is defined by

$$\operatorname{diam}(A) := \sup\{d(x, y) \mid x, y \in A\}.$$

Then A is bounded if and only if $\operatorname{diam}(A) < \infty$.

Remark 1.2.16. *The notion of boundedness depends on the metric which is considered. For any given metric space (X, d),* every *subset $A \subset X$ is bounded with respect to the new metric*

$$(x, y) \mapsto \frac{d(x, y)}{1 + d(x, y)} \ .$$

Definition 1.2.17. *Let (X, d) be a metric space. For two subsets $A, B \subset X$ the* ***distance from** A **to** B* *is the number*

$$d(A, B) := \inf_{x \in A} \inf_{y \in B} d(x, y) \ .$$

If $A = \{x\}$ then $d(x, B) := d(\{x\}, B)$ is the ***distance from** x **to the set** B.*

Remark 1.2.18. *1) Let A be a non-empty subset of a metric space X. We have seen in Section 1.1 that (A, d_A) is a metric space, where d_A is the induced metric. Let $B(x, \varepsilon)$ be an open ball in X and let $B^A(x, \varepsilon)$ the open ball in A with radius $\varepsilon > 0$ and center $x \in X$, i.e.,*

$$B^A(x, \varepsilon) = \{y \in A : \ d(x, y) < \varepsilon\},$$

we see that $B^A(x, \varepsilon) = B(x, \varepsilon) \cap A$. A similar relation holds for closed and open sets in A. A subset $O \subset A$ is open in A (i.e., with respect to d_A) if and only if $O = A \cap U$ where U is a open set in X. Similarly, a subset $B \subset A$ is closed in A if and only if $B = A \cap F$ where F is a closed set in X.

2) Observe that if (X, d) is a metric space, then X may or may not be bounded in its own right: think of the intervals $(0, 1)$ and $(0, \infty)$ with respect to the canonical distance. If the metric space is bounded, then so are all its subsets; more generally, subsets of bounded sets are clearly bounded.

3) The closure of a bounded set is bounded (try to prove it!) and hence, in particular, bounded sets always have bounded boundary – the converse is obviously wrong for any unbounded metric space: think of the example of the unbounded set $\mathbb{R}^n \setminus B(x, r)$, for any given $x \in \mathbb{R}^n$ and $r > 0$, whose boundary is the sphere

$$S(0, 1) := \{y \in \mathbb{R}^n : d(x, y) = r\} \ .$$

4) Let A, B be two subsets of a metric space (X, d). If $A \cap B \neq \emptyset$, then $d(A, B) = 0$. The converse is in general not true. Take for instance $X = \mathbb{R}$ with the absolute-value metric and the sets $A = (0, 1)$ and $B = (1, 2)$.

1.2.1 Exercises

E 1.2.19. *Let X be a metric space and let $x \in X$. Prove that the set $\{x\}$ is closed.*

E 1.2.20. *Let (X, d_X) be a metric space.*

$a)$ *For d_X being the discrete metric, show that X is the only dense set.*

$b)$ *Show that $A \subset X$ is dense if and only if $A \cap B_r \neq \emptyset$ for all open balls $B_r(x)$, $x \in X$.*

E 1.2.21. *Let (X, d) be a metric space and let A be a non-empty subset of it. Prove that:*

$i)$ *If $x \in A$ then $d(x, A) = 0$.*

$ii)$ *If* $d(x, A) = 0$ *then* $x \in \overline{A}$.

Conclude that $d(x, A) = 0$ *if and only if* $x \in \overline{A}$.

E 1.2.22. *Prove that the union of two bounded subsets* A *and* B *of a metric space* (X, d) *is bounded.*

E 1.2.23. *Let* (X, d) *be a metric space. Let* A, B *be two non-empty subsets of* X. *Show that*

$i)$ *diam*$(\overline{A}) =$ *diam*(A).

$ii)$ *Assume that* $A \cap B \neq \emptyset$. *Then* $d(A, B) = 0$.

$iii)$ *diam*$(A \cup B) \leq$ *diam*$(A) + d(A, B) +$ *diam*(B).

E 1.2.24. *A metric* d *on a vector space* X *is called an* ***ultrametric*** *if for all* x, y *and* $z \in X$

$$d(x, z) \leq \max\{d(x, y), d(y, z)\},$$

and we say that (X, d) *is an* ***ultrametric space***. *The discrete metric is for example an ultrametric (but general discrete metric spaces are not necessarily ultrametric spaces!).*

Let (X, d) *be a ultrametric space.*

1. *Show that for each* $x, y, z \in X$ *with* $d(x, y) \neq d(y, z)$ *we have*

$$d(x, z) = \max\{d(x, y), d(y, z)\}.$$

2. *Show that each open ball* $B(x, r)$ *is closed and open. Moreover,* $B(y, r) = B(x, r)$ *for all* $y \in B(x, r)$.

3. *Show that each closed ball* $\overline{B}(x, r)$ *is closed and open and moreover,* $\overline{B}(y, r) = \overline{B}(x, r)$ *for all* $y \in B(x, r)$.

1.3 Sequences in metric spaces

A **sequence** in a set X is a mapping $\mathbb{N} \ni n \mapsto x_n \in X$, usually denoted by $(x_n)_{n\in\mathbb{N}}$. As soon as the set X carries suitable structures of topological nature, we may inquire convergence of sequences.

1.3.1 Convergence

Definition 1.3.1. *We say that a sequence* $(x_n)_{n\in\mathbb{N}} \subset X$ ***converges*** *to some element* $x \in X$ *– and we write* $\lim_{n\to\infty} x_n = x$, $x_n \to x$ *as* $n \to \infty$, *or simply* $x_n \to x$ *– if for every* $\varepsilon > 0$ *there exists* $n_0 \in \mathbb{N}$ *such that for every* $n \geq n_0$ *one has* $d(x_n, x) \leq \varepsilon$. *If* $x_n \to x$ *then we call* x *the* ***limit*** *of* $(x_n)_{n\in\mathbb{N}}$.

Observe that this agrees with the classical definition of sequence convergence in normed space (and, in particular, in $\mathbb{R}$) which you have learned in the course 1144.

Remark 1.3.2. *Let* (X, d) *be a metric space.*

1) *A sequence* $(x_n)_{n\in\mathbb{N}} \subset X$ *converges to some* $x \in X$ *if and only if the sequence of real numbers* $(d(x_n, x))_{n\in\mathbb{N}}$ *converges to 0 as* $n \to \infty$. *Equivalently,* $x_n \to x$ *if and only if for every neighborhood* U *of* x *there exists* $n_0 \in \mathbb{N}$ *such that for every* $n \geq n_0$ *one has* $x_n \in U$.

2) *Each convergent sequence on X has a unique limit. This is, if $(x_n)_{n\in\mathbb{N}} \subset X$ is a sequence and $x, y \in X$ such that $x_n \to x$ and $x_n \to y$ as $n \to \infty$, then $x = y$. This is due to the fact that each metric space is a Hausdorff space.*

Example 1.3.3. *Consider the metric space $(C([0,1]), d_\infty)$, see Example 1.1.3. Then a sequence $(f_n)_{n\in\mathbb{N}} \subset C[0,1]$ converges to $f \in C[0,1]$ if and only if the sequence $(f_n)_{n\in\mathbb{N}}$ converges uniformly to f in the sense that you have learned in the course 1144.*

Example 1.3.4. *Let X be a non-empty set and d be the discrete metric upon it. If a sequence $(x_n)_{n\in\mathbb{N}} \subset X$ converges to x with respect to d, then there exists $n_0 \in \mathbb{N}$ such that for all $n \geq n_0$ on has $\mathrm{d}(x_n, x) \leq \frac{1}{2}$. Since d is the discrete metric this means that $\mathrm{d}(x_n, x) = 0$ and thus $x_n = x$ for all $n \geq n_0$: we have thus observed that every convergent sequence with respect to the discrete metric is eventually constant. This result clearly holds in the context of general discrete metric spaces. Conversely, every eventually constant function is convergent with respect to any metric.*

Since every metric defines a topology in a natural way, metric spaces form a subclass of the topological spaces. In general topological spaces it might be difficult to characterize the closure or the boundary of a given subset $A \subset X$. For metric spaces, however, it is always possible to characterize closed sets of X in terms of convergent sequences as follows.

Theorem 1.3.5. *Let A be a subset of a metric space X. Then the following assertions are equivalent.*

a) $x_0 \in \overline{A}$.

b) There exists a sequence $(x_n)_{n\in\mathbb{N}} \subset X$ such that $x_n \to x_0$.

Proof. In order to show that $a) \Rightarrow b)$, let $x_0 \in \overline{A}$. Then for every $n \in \mathbb{N}$ one has $B(x_0, \frac{1}{n}) \cap A \neq \infty$. Thus for every $n \in \mathbb{N}$ we can find $x_n \in A$ such that $d(x_0, x_n) \leq \frac{1}{n}$. This implies that the sequence $x_n \to x_0$.

Conversely, assume that $b)$ holds and let $(x_n)_{n\in\mathbb{N}} \subset X$ which converges to some $x_0 \in X$. Let $\varepsilon > 0$. From the convergence there exists $n_0 \in \mathbb{N}$ such that $d(x_n, x_0) \leq \varepsilon$ for all $n \geq n_0$. This implies that $B(x_0, \frac{1}{n}) \cap A \neq \infty$. Because ε is arbitrary, we conclude that $x_0 \in \overline{A}$. $\square$

We thus deduce the following characterizations.

Corollary 1.3.6. *A subset A of a metric space X is closed if and only if for every sequence $(x_n)_{n\in\mathbb{N}} \subset A$ converging to some $x_0 \in X$, one has $x \in A$.*

Corollary 1.3.7. *Let A be a subset of a metric space X. Then the following assertions are equivalent.*

a) A is dense in X.

b) For each $x_0 \in X$ there exists $(x_n)_{n\in\mathbb{N}} \subset A$ such that $x_n \to x_0$.

c) $A \cap O \neq \emptyset$ for any open subset $O \subset X$.

1.3.2 Cauchy sequences and complete metric spaces

Let (X, d) be a metric space. A sequence $(x_n)_{n\in\mathbb{N}} \subset X$ is a **Cauchy sequence** if for all $\varepsilon > 0$ there exists $n_0 \in \mathbb{N}$ such that

$$d(x_n, x_m) \leq \varepsilon \text{ for all } n, m \geq n_0.$$

By the triangle inequality, each convergent sequence in a metric space X is a Cauchy sequence. The converse is however not true. For instance, consider $(0,1)$ with the absolute-value metric and the sequence $x_n := 1 - \frac{1}{n} \in (0,1)$. Since $(x_n)_{n\in\mathbb{N}}$ converges in $\mathbb{R}$ to 1, $(x_n)_{n\in\mathbb{N}}$ is a Cauchy sequence (in $\mathbb{R}$ and then) in $(0,1)$ but $(x_n)_{n\in\mathbb{N}}$ has no limit in $(0,1)$.

Definition 1.3.8. *We say that a metric space (X,d) is **complete** if every Cauchy sequence is convergent.*

For complete metric spaces, like the Euclidean spaces $\mathbb{R}^n$, the definition of Cauchy sequences is thus a convergence criterion. This criterion does not involve the limit of the sequence, hence it gives us a tool to prove that a sequence is convergent without knowing its limit.

Example 1.3.9. *1) $(\mathbb{R}^n, d_2)$ and $(C[0,1], d_\infty)$ are complete metric spaces*

2) Every set endowed with the discrete metric is complete. In fact, let $(x_n)_{n\in\mathbb{N}} \in X$ be a Cauchy sequence. Then there exists $n_0 \in \mathbb{N}$ such that for all $n, m \geq n_0$ on has $d(x_n, x_m) \leq \frac{1}{2}$ and therefore $x_n = x_m$, thus $(x_n)_{n\in\mathbb{N}}$ is eventually constant and hence convergent. The same proof shows that also general discrete metric spaces are complete.

Theorem 1.3.10. *Let (X,d) be a metric space. Then the following assertions hold.*

1) If (X,d) is complete, then also every closed subset $F \subset X$ is complete with respect to the induced metric.

2) If $A \subset X$ is complete, then A is closed.

Proof. 1) Let $(x_n)_{n\in\mathbb{N}} \subset F$ be a Cauchy sequence. Then $(x_n)_{n\in\mathbb{N}}$ converges to some element $x \in X$ and since F is closed, $x \in F$. Thus F is complete.

2) Let $(x_n)_{n\in\mathbb{N}} \subset A$ converging to $x_0 \in X$. Thus is a Cauchy sequence in A and then in X. Since A is complete, $(x_n)_{n\in\mathbb{N}}$ converges to a an element y belonging to A. By uniqueness of the limit, $x_0 = y$ and thus $x_0 \in F$. By Corollary 1.3.6 this implies that F is closed. ☐

Let $([a_n, b_n])_{n\in\mathbb{N}}$ be a sequence of closed intervals such that the sequences $(a_n)_{n\in\mathbb{N}}$ and $(b_n)_{n\in\mathbb{N}}$ are adjacent, i.e., $(a_n)_{n\in\mathbb{N}}$ is increasing and $(b_n)_{n\in\mathbb{N}}$ decreasing with $(b_n - a_n) \to 0$. Then it is known that $\bigcap_{n\in\mathbb{N}}[a_n, b_n] = \{l\}$ with $l := \lim\limits_{n\to\infty} a_n = \lim\limits_{n\to\infty} b_n$. For a complete metric space (X,d) a similar result holds.

Proposition 1.3.11. *Let $(F_n)_{n\in\mathbb{N}}$ be a decreasing sequence of closed subset of a complete metric space (X,d) such that $\operatorname{diam}(F_n) \to 0$ as $n \to \infty$. Then $\bigcap_{n\in\mathbb{N}} F_n = \{x\}$ for some $x \in X$.*

Proof. Since $\operatorname{diam}(F_n) \to 0$ the intersection $F := \bigcap_{n\in\mathbb{N}} F_n$ contains at most one element. Regarding existence, we choose $x_n \in F_n$ for each $n \in \mathbb{N}$. Since

$$d(x_n, x_m) \leq \operatorname{diam}(F_N) \text{ for all } n, m \geq N$$

the sequence $(x_n)_{n\in\mathbb{N}}$ is a Cauchy sequence and thus converges to some element $x \in X$. For every $k \in \mathbb{N}$, we have $x_n \in F_n \subset F_k$ for $n \geq k$. Since F_k are closed, the limit x of $(x_n)_{n\in\mathbb{N}}$ belongs to F_k. Thus F is not empty. ☐

1.3.3 Subsequences and cluster points

As introduced above, a sequence $(x_n)_{n\in\mathbb{N}}$ on a metric space (X,d) is a mapping $n \mapsto x_n$ from $\mathbb{N}$ to X. A **subsequence** of $(x_n)_{n\in\mathbb{N}}$ is a mapping $k \mapsto x_{n_k}$ from $\mathbb{N}$ to X where $k \mapsto n_k : \mathbb{N} \to \mathbb{N}$ is strictly increasing. For example, $(x_{n+1})_{n\in\mathbb{N}}$ is a subsequence of $(x_n)_{n\in\mathbb{N}}$ because $n \mapsto n+1$ is strictly increasing from $\mathbb{N}$ into $\mathbb{N}$.

Remark 1.3.12. *Bijectivity of $k \mapsto n_k$ is a necessary but not sufficient condition for $(x_{n_k})_{k\in\mathbb{N}}$ to be a subsequence of $(x_n)_{n\in\mathbb{N}}$: e.g.,*

$$x_1, x_4, x_2, x_5, x_6, x_7, \ldots$$

is not a subsequence of $(x_n)_{n\in\mathbb{N}}$.

Proposition 1.3.13. *If a sequence $(x_n)_{n\in\mathbb{N}} \subset X$ converges to some $x \in X$, then every subsequence converges to the same limit x.*

Proof. Let $\varepsilon > 0$. Since $(x_n)_{n\in\mathbb{N}}$ converges to x, there exists $n_0 \in \mathbb{N}$ such that $d(x, x_n) \leq \varepsilon$ if $n \geq n_0$. Taking $k_0 \in \mathbb{N}$ such that $n_{k_0} \geq n_0$ we obtain that $d(x, x_{n_k}) \leq \varepsilon$. □

Note that the converse of the statement of Proposition 1.3.13 is also true since $(x_n)_{n\in\mathbb{N}}$ is always a subsequence of itself.

Definition 1.3.14. *We say that $x \in X$ is a* ***cluster point*** *or* ***accumulation point*** *of the sequence $(x_n)_{n\in\mathbb{N}}$ if there exists a subsequence $(x_{n_k})_{k\in\mathbb{N}}$ converging to x.*

For example the real sequence

$$x_n := \begin{cases} \frac{1}{n} & \text{if } n = 2k,\ k \in \mathbb{N}, \\ 1 & \text{otherwise}, \end{cases}$$

has two cluster points. Indeed, the subsequences $(x_{2n})_{n\in\mathbb{N}}$ and $(x_{2n+1})_{n\in\mathbb{N}}$ converges to 0 and 1 respectively. Clearly, if x is the limit of a convergent sequence $(x_n)_{n\in\mathbb{N}}$, then x is the unique cluster point of $(x_n)_{n\in\mathbb{N}}$.

Proposition 1.3.15. *Let $(x_n)_{n\in\mathbb{N}}$ be a sequence in a metric space (X, d). Then the following assertions are equivalent.*

a) x is a cluster point of $(x_n)_{n\in\mathbb{N}}$.

b) For all $\varepsilon > 0$ and $n \in \mathbb{N}$ there exists $m > n$ such that $d(x_m, x) \leq \varepsilon$.

c) For each $p \in \mathbb{N}$, $x \in \overline{\{x_n,\ n \geq p\}}$.

Proof. The implication $a) \Rightarrow b)$ is clear. Indeed, let $\varepsilon > 0$ and $n \in \mathbb{N}$ be fixed. Since there exists $k(\varepsilon) \in \mathbb{N}$ such that $d(x_{n_k}, x) \leq \varepsilon$ for each $k \geq k(\varepsilon)$, it suffices to take $m = n_k$ for $k \geq (\varepsilon)$ with $n_k \geq n$.

The implication $b) \Rightarrow c)$ is left as an exercise.

Assume now that $c)$ holds. For every $k, n \in \mathbb{N}$ there exists $n(k) \in \mathbb{N}$ such that $d(x, x_{n(k)}) \leq 1/k$. We can then define a subsequence $(x_{n_k})_{k\in\mathbb{N}}$ such that x_{n_k} is the smallest integer with

$$x_{n_{k-1}} < x_{n_k} \quad \text{and} \quad d(x, x_{n_k}) \leq 1/k.$$

This subsequence converges to x, which shows that $a)$ holds. □

Remark 1.3.16. *Let (X, d) be a metric space. Then x is a cluster point of a sequence $(x_n)_{n\in\mathbb{N}} \subset X$ if and only if every open ball $B(x, \varepsilon)$ contains x_n for infinitely many $n \in \mathbb{N}$.*

In the previous section we have seen that, in general metric spaces, not every Cauchy sequence converges. It turns out that convergent Cauchy sequences can indeed be characterized in terms of cluster points.

Proposition 1.3.17. *If a Cauchy sequence $(x_n)_{n\in\mathbb{N}}$ has a cluster point x, then $(x_n)_{n\in\mathbb{N}}$ converges and x is its limit.*

Proof. Let $\varepsilon > 0$. Since x is a cluster point there exists $k_0 \in \mathbb{N}$ such that $d(x, x_{n_k}) \leq \varepsilon/2$ for $k \geq k_0$. Since $(x_n)_{n\in\mathbb{N}}$ is a Cauchy sequence there exists $n_0 \in \mathbb{N}$ such that $d(x_n, x_m) \leq \varepsilon/2$ for $n, m \geq n_0$. By taking $n_1 := max(n_0, n_{k_0})$ and $k \geq k_0$ such that $n_k \geq n_0$ we obtain

$$d(x, x_m) \leq d(x, x_{n_k}) + d(x_{n_k}, x_m) \leq \varepsilon/2 + \varepsilon/2 = \varepsilon$$

for all $m \geq n_1$. Thus $x_n \to x$. □

1.3.4 Exercises

E 1.3.18. *Show that each convergent sequence on a metric space X has a unique limit.*

E 1.3.19. *Let X be a metric space and let $x \in X$. Using Corollary 1.3.6 show that the set $\{x\}$ is closed.*

E 1.3.20. *Let (X, d) be a metric space and let $(x_n)_{n\in\mathbb{N}}$ be a sequence of elements of X. Then (x_n) is a Cauchy sequence if and only if $\lim\limits_{n\to\infty} \operatorname{diam}(A_n) = 0$ where $A_n := \{x_n, x_{n+1}, ...\}$.*

E 1.3.21. *Let (X, d) be a metric space. Show that X consists of only one element if and only if every bounded sequence is convergent.*

E 1.3.22. *Consider the vector space*

$$\ell^1(\mathbb{N}) := \{(x_n)_{n\in\mathbb{N}} : \sum_{n=0}^{\infty} |x_n| < \infty\}$$

of all absolutely convergent series of real numbers and the vector space c_{00} introduced in Exercise 1.1.7. Clearly $c_{00}(\mathbb{N}) \subset \ell^1(\mathbb{N})$ and we can define on both $\ell^1(\mathbb{N}) \times \ell^1(\mathbb{N})$ and $c_{00}(\mathbb{N}) \times c_{00}(\mathbb{N})$ the mapping

$$d(x, y) := \sum_{n=1}^{\infty} |x_n - y_n| \,.$$

Show that

1. *$(\ell^1(\mathbb{N}), d)$ is a complete metric space.*
2. *$c_{00}(\mathbb{N})$ is dense in $\ell^1(\mathbb{N})$, i.e., $\overline{c_{00}} = \ell^1(\mathbb{N})$.*

1.4 Continuous functions

The purpose of this paragraph is to introduce a general concept of continuous functions on metric spaces. For this let (X, d_X) and (Y, d_Y) be two metric spaces.

Definition 1.4.1. *A function $f : X \to Y$ is said to be* ***continuous at*** *$x_0 \in X$ if for all $\varepsilon > 0$ there exists $\delta = \delta(x_0, \varepsilon)$ such that*

$$d_X(x_0, x) \leq \delta \Rightarrow d_Y(f(x_0), f(y)) \leq \varepsilon.$$

The function f is ***continuous on*** *X if it is continuous at every point $x \in X$.*

One readily sees that a function $f : X \to Y$ is continuous at $x_0 \in X$ if for every neighborhood U of $f(x_0) \in Y$ the set $f^{-1}(U)$ is a neighborhood of x_0.

Theorem 1.4.2. *Let $f : X \to Y$. The following assertions are equivalent.*

a) f is continuous at $x_0 \in X$.

*b) f is **sequentially continuous at** x_0, that is for every sequence $(x_n)_{n\in\mathbb{N}} \subset X$ which converges to x_0 one has $\lim\limits_{n\to\infty} f(x_n) = f(x_0)$.*

Proof. For the implication $a) \Rightarrow b)$ let $(x_n)_{n\in\mathbb{N}} \subset X$ be convergent to x_0. For each $\varepsilon > 0$ there exists $\delta > 0$ such that $d_Y(f(x_0), f(y)) \le \varepsilon$ whenever $d_X(x_0, x) \le \delta$. By the convergence, there exists $n_0 \in \mathbb{N}$ such that $d_X(x_0, x_n) \le \delta$. Then $d_Y(f(x_0), f(x_n)) \le \varepsilon$ for all $n \ge n_0$. Hence $f(x_n) \to f(x_0)$ as $n \to \infty$.

For the converse implication we assume that f is sequentially continuous at x_0. If f is not continuous, then there exists $\varepsilon > 0$ such that for all $n \in \mathbb{N}$ we can find $x_n \in X$ such that $d_X(x_0, x_n) \le \frac{1}{n}$ with $d_Y(f(x_0), f(x_n)) \ge \varepsilon$. We obtain a sequence $(x_n)_{n\in\mathbb{N}}$ which converges to x_0. Since f is sequentially continuous we have $(f(x_n))_{n\in\mathbb{N}}$ converges to $f(x_0)$, contradicting the fact that $d_Y(f(x_0), f(x_n)) \ge \varepsilon$ for every $n \in \mathbb{N}$. Therefore f is continuous at x_0. □

Example 1.4.3. *Any metric $d : X \times X \to \mathbb{R}$ on X is a continuous function. In fact, let $(x_n)_{n\in\mathbb{N}}$ and $(y_n)_{n\in\mathbb{N}} \subset X$ be such that $x_n \to x$ and $y_n \to y$ as $n \to \infty$. From inequality (1.2) we have*

$$|d(x_n, y_n) - d(x, y)| \le d(x_n, x) + d(y_n, y) \text{ for all } n \in \mathbb{N}.$$

Thus $\lim\limits_{n\to\infty} d(x_n, y_n) = d(x, y)$ and Theorem 1.4.2 implies that d is continuous.

Continuous functions can be defined also between topological spaces which are not necessarily metric spaces. The following proposition shows that, for metric spaces, Definition 1.4.1 is actually equivalent to the more general definition which is also valid for maps between topological spaces.

Proposition 1.4.4. *Let X and Y be metric spaces and let $f : X \to Y$ be given. The following assertions are equivalent.*

a) f is continuous.

b) For every open set $O \subset Y$ one has $f^{-1}(O)$ is open.

c) For every closed set $F \subset Y$ one has $f^{-1}(F)$ is closed.

Proof. It suffices to proof that a) is equivalent to b). First assume that f is continuous and let $O \subset X$ be open. Let $x \in f^{-1}(O)$ be arbitrary. Then $f(x_0) \in O$. Since O is open there exists $\varepsilon > 0$ such that $B(f(x_0), \varepsilon) \subset O$. Because f is continuous, there exists $\delta > 0$ such that for every $x \in B(x_0, \delta)$ one has $f(x) \in B(f(x_0), \varepsilon) \subset O$. Thus $B(x_0, \delta) \subset f^{-1}(B(f(x_0), \varepsilon)) \subset f^{-1}(O)$. Therefore, $f^{-1}(O)$ is open.
Conversely, assume that (b) holds. Let $x_0 \in X$ and let V be a neighborhood of $f(x_0)$. There exists an open ball $B(y, r)$ such that $f(x_0) \in B(y, r) \subset V$. Then $x_0 \in f^{-1}(B(y, r)) \subset f^{-1}(V)$. Since $f^{-1}(B(y, r))$ is open there exists $r' > 0$ such that $x_0 \in B(x_0, r') \subset f^{-1}(B(y, r))$. This implies that $f^{-1}(V)$ is a neighborhood of x_0. □

Example 1.4.5. *Let (X, d_X) be a metric space with the discrete metric. Then every function f from X to a metric space Y is continuous.*

Remark 1.4.6. *For a continuous function f between two metric spaces (X, d_X) and (Y, d_Y), the image $f(O)$ of a open set $O \subset X$ is not necessary open in Y. Also the image $f(F)$ of a closed set $F \subset X$ is not necessary closed in Y. Take for example $f : \mathbb{R} \to \mathbb{R}$ given by $f(x) = x^2$. Then for the open set $]-1, 1[$, the image $f(]-1, 1[) = [0, 1[$ is not open in $\mathbb{R}$.*

Definition 1.4.7. *Let $(X, d_X), (Y, d_Y)$ be two metric spaces. Then $f : X \to Y$ is said to be*

*1) **uniformly continuous** if for all $\varepsilon > 0$ there exists $\delta > 0$ such that*

$$d_Y(f(x), f(y)) \leq \varepsilon \qquad \text{for all } x, y \in X \text{ s.t. } d(x, y) < \delta;$$

*2) **Lipschitz continuous** with **Lipschitz constant** $L > 0$ if*

$$d_Y(f(x), f(y)) \leq L\, d_X(x, y) \, \text{for all } x, y \in X \; ;$$

*3) a **strict contraction** if it is Lipschitz continuous with constant $L \in [0, 1)$.*

It is important to note that, in the definition of uniform continuity, δ does *not* depend on the point of continuity. This means that, unlike the continuity which is a local property, the uniform continuity is a global property. Note that each uniformly continuous function is continuous and each Lipschitz continuous function is uniformly continuous.

Proposition 1.4.8. *Let (X, d) be a metric space and let $A \subset X$. Then the function $x \mapsto d(x, A)$ is Lipschitz continuous.*

Proof. Let $x, y \in X$. Then for $a \in A$

$$d(x, A) \leq d(x, a) \leq \; d(x, y) + d(y, a).$$

Thus

$$d(x, A) \leq d(x, a) \leq \; d(x, y) + \inf_{a \in A} d(y, a).$$

Similarly, we obtain

$$d(y, A) \leq d(x, y) + \inf_{a \in A} d(x, a).$$

Finally

$$|d(x, A) - d(y, A)| \leq d(x, y) \; ,$$

which is what wanted to prove. □

As a corollary we obtain the following characterization of the closure $\overline{A}$ of a set $A \subset X$.

Corollary 1.4.9. *Let X be a metric space and let $A \subset X$. Then*

$$\overline{A} = \{x \in X : \; d(x, A) = 0\}.$$

Proof. Since $x \mapsto d(x, A)$ is continuous, the set $B := \{x \in X : \; d(x, A) = 0\}$ is closed. Since $A \subset B$, it follows that B contains $\overline{B}$. Conversely, let $x \in B$. For every $n \in \mathbb{N}$ there exists $x_n \in A$ such that

$$d(x, x_n) \leq \frac{1}{n}.$$

Thus there exists a sequence $(x_n)_{n \in \mathbb{N}} \subset A$ which converges to x and then $x \in \overline{A}$. This shows that $B \subset \overline{A}$. □

1.4.1 Exercises

E 1.4.10. *Let (X, d_X) and (Y, d_Y) be two metric spaces. Show that a function $f : X \to Y$ is continuous at $x \in X$ if and only if for each neighbourhood V of $f(x)$ there exists a neighbourhood U of x such that $f(U) \subset V$.*

E 1.4.11. *Let (X, d_X) be a metric space with the discrete metric. Show that every function f from X to a metric space Y is continuous.*

E 1.4.12. *Prove that $f : \mathbb{R} \to \mathbb{R} : x \mapsto x^2$ is continuous but not uniformly continuous.*

E 1.4.13. *Let (X, d_X) and (Y, d_Y) be two metric spaces and (Y, d_Y) complete. Suppose that $f : X \to Y$ is bijective and uniformly continuous and that f^{-1} is continuous. Show that (X, d_X) is also complete.*

1.5 Extension by density of continuous functions

Let (X, d_X) and (Y, d_Y) be two metric spaces and D be a subset of X. Let $f : D \to Y$ and $g : X \to Y$. We say that g **extends** f, or that f is an **extension** of g, if

$$g(x) = f(x) \quad \text{for every } x \in D.$$

Proposition 1.5.1. *Let f, g be two continuous functions from X to Y. Let D be a dense subset of X. Then the following assertions hold.*

1) If $f(x) = g(x)$ for every $x \in D$, then $f(x) = g(x)$ for all $x \in X$.

2) Assume that $Y = \mathbb{R}$ and that $f(x) \le g(x)$ for every $x \in D$, then $f(x) \le g(x)$ for all $x \in X$.

Proof. 1) Let $x \in X$ and let $(x_n)_{n\in\mathbb{N}} \subset D$ converge to x in X. Since f and g are continuous, we have $f(x_n) \to f(x)$ and $g(x_n) \to g(x)$. We obtain

$$d_Y(f(x), g(x)) \le d_Y(f(x), f(x_n)) + d_Y(f(x_n), g(x_n)) + d_Y(g(x_n), g(x))$$

Letting $n \to \infty$, we conclude that $d_Y(f(x) - g(x)) \le 0$ and thus $f(x) = g(x)$. This completes the proof. An analogous argument shows 2). □

Proposition 1.5.1, shows that an extension of a continuous function on a dense subset (if it exists) is unique. Indeed, if f is defined on a dense subset $D \subset X$ and if g and h are two extensions of f, then we have $f(x) = g(x) = h(x)$ for every $x \in D$. By Proposition 1.5.1, this implies $g(x) = h(x)$ for all $x \in X$.
To prove existence of an extension we need additional conditions as formulated in the following theorem.

Theorem 1.5.2. *Let f be a uniformly continuous function from a dense subset D of a metric space (X, d_X) to a complete metric (Y, d_Y). Then f has a unique continuous extension $g : X \to Y$. Furthermore, $g : X \to Y$ is uniformly continuous.*

Proof. Let $x \in X$. Since D is dense there exists a sequence $(x_n)_{n\in\mathbb{N}} \subset D$ which converges to x in X Because f is uniformly continuous, $(f(x_n))_{n\in\mathbb{N}}$ is a Cauchy sequence in Y (observe that mere continuity of f would not do the job!) and hence $\lim\limits_{n\to\infty} f(x_n) =: g(x)$ exists since Y is complete. The element $g(x) \in Y$ is independent of the sequence $(x_n)_{n\in\mathbb{N}}$ which converges to x : if $(x'_n)_{n\in\mathbb{N}}$

is another sequence with $x'_n \to x$ and $\hat{g}(x)$ is the limit of the Cauchy sequence $(f(x'_n))_{n\in\mathbb{N}} \subset Y$ then, using the uniform continuity of f, we see that $\hat{g}(x) = g(x)$. Hence, the function $g : X \to Y$ given by $g(x) = \lim_{n\to\infty} f(x_n)$ is defined and is an extension of f since $g(x) = f(x) = \lim_{n\to\infty} f(x_n)$ for each $x \in D$.
It remains to show that g is uniformly continuous. Let $\varepsilon > 0$. Since f is uniformly continuous on D there exists $\delta_\varepsilon > 0$ such that $d_Y(f(x), f(y)) \le \varepsilon$ for every $x, y \in D$ with $d_X(x, y) \le \delta_\varepsilon$. On the other hand, there exist two sequence $(x_n)_{n\in\mathbb{N}}$ and $(y_n)_{n\in\mathbb{N}}$ of elements of D such that $x_n \to x$ and $y_n \to y$. Then there exists $n_0(\delta) \in \mathbb{N}$ such that for every $n \ge n_0(\delta)$ and for $\|x - y\|_X \le \delta_\varepsilon/3$ one has

$$d_X(x_n, y_n) \le d_X(x_n, x) + d_X(x, y) + d_X(y_n, y) \le \delta_\varepsilon/3 + \delta_\varepsilon/3 + \delta_\varepsilon/3 = \delta_\varepsilon.$$

We deduce that

$$\begin{aligned} d_Y(g(x), g(y)) &\le d_Y(g(x), f(x_n)) + d_Y(f(x_n), g(y_n)) + d_Y(g(y_n), g(y)) \\ &\le d_Y(g(x), f(x_n)) + \varepsilon + d_Y(g(y_n), g(y)) \\ &\le \varepsilon \quad (\text{ by letting } n \to \infty) \end{aligned}$$

when $d_X(x, y) \le \delta_\varepsilon/3$. □

1.6 Separable metric spaces

While we know that the set of real numbers is not countable, several results about continuous functions of one real variable can be reduced to studying their properties whenever restricted to the set of rational numbers. This is of uttermost importance because $\mathbb{Q}$ is of course much smaller than $\mathbb{R}$ and, more importantly, because working along sequences of numbers often allows for constructive proofs that can be turned into algorithms. Clearly, the property of containing a countable dense subset is not limited to $\mathbb{R}$ or $\mathbb{C} = \mathbb{R} + i\mathbb{R}$, as the set of polynomials with real vs. rational coefficients show.

Definition 1.6.1. *A metric space is called* ***separable*** *if it contains a countable dense subset.*

In other words, a metric space (X, d) is separable if there exists a countable set $D := \{x_n : n \in \mathbb{N}\}$ such that for any $x \in X$ and each $\varepsilon > 0$

$$B(x, \varepsilon) \cap D \ne \emptyset.$$

Example 1.6.2. *1) As already suggested above, $\mathbb{R}$ is separable, since $\mathbb{Q}$ is dense. For the same reason also $\mathbb{C}$ and $\mathbb{R}^n, \mathbb{C}^n$ are separable.*

2) Let X be a non-empty set and d be the discrete metric on it. In order for a subset D to be dense in X, each point of X should be the limit of a suitable sequence $(x_n)_{n\in\mathbb{N}} \subset D$; but we have seen in Example 1.3.4 that convergent sequences with respect to the discrete metric are necessarily eventually constant, i.e., such a sequence $(x_n)_{n\in\mathbb{N}}$ cannot converge to any element outside D. Accordingly, the only dense subset of X is X itself, and we conclude that (X, d) is separable if and only if X if countable. In particular, $\mathbb{R}$ is not separable whenever endowed with the discrete metric.

Proposition 1.6.3. *Each subset of a separable metric space is separable.*

Proof. Let (X, d) be a metric space and let $Y \subset X$ be a non-empty subset. Assume that X is separable and let $D := \{a_n : n \in \mathbb{N}\}$ be dense in X. For each pair $(a_k, n) \subset D \times \mathbb{N}$ choose $y_{nk} \in Y$ such that

$$d(a_k, y_{nk}) \le d(a_k, Y) + \frac{1}{n}.$$

We will show now that y_{nk} forms a countable dense subset of Y. For this, let $y \in Y$ and $\varepsilon > 0$. By the separability of X, there exists $a_{k_0} \in D$ such that

$$d(y, a_{k_0}) < \frac{\varepsilon}{3}.$$

Next choosing $\frac{1}{n_0} < \frac{\varepsilon}{3}$, we obtain

$$\begin{aligned}
d(y, y_{k_0 n_0}) &\le d(y, a_{k_0}) + d(a_{k_0}, y_{k_0 n_0}) \\
&\le d(y, a_{k_0}) + d(a_{k_0}, Y) + \frac{1}{n_0} \\
&\le d(y, a_{k_0}) + d(a_{k_0}, y) + \frac{1}{n_0} \qquad \text{(since } y \in Y\text{)} \\
&\le \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} = \varepsilon.
\end{aligned}$$

This completes the proof. □

Proposition 1.6.4. *A metric space is separable if and only if there exists a countable family $\{O_n : n \in \mathbb{N}\}$ of open set of X such that for every open set O of X and for every $x \in X$ there exists O_{n_0} such that $x \subset O_{n_0} \subset O$.*

Proof. Assume that X is separable and let $D = \{a_n : n \in \mathbb{N}\}$ be dense in X. Let $O \subset X$ be open and take $x \in O$. Set $\delta := d(x, O^{\complement}) = \inf_{y \in O^{\complement}} d(x, y)$ and let $m \in \mathbb{N}$ be such that $m > 2/\delta$. Since D is dense in X, there exists $a_n \in D$ with

$$d(x, a_n) < 1/m.$$

Moreover, the open ball $B(a_n, 1/m) \subset O$. Otherwise, there would exists $z \in O^{\complement}$ such that

$$d(a_n, z) < 1/m,$$

and we would then obtain that

$$d(x, z) \le d(x, a_n) + d(a_n, z) < 1/m + 1/m = 2/m < \delta,$$

which is absurd. We conclude then that the open balls $B(a_n, 1/m)$ satisfy the necessary condition of the proposition. Conversely, assume that such open set $O_n, n \in \mathbb{N}$ exists and take $a_n \in O_n$ for each $n \in \mathbb{N}$. Then the countable set $\{a_1, a_2, ..., a_n, ...\}$ is dense in X. In fact, for each $x \in X$ and for each open ball $B(x, \varepsilon)$ there exists O_n such that $x \in O_n \subset B(x, \varepsilon)$. Thus $a_n \in O_n \cap B(x, \varepsilon)$. □

1.6.1 Exercises

E 1.6.5. *Suppose that (X, d_X) is a separable metric space and (Y, d_Y) a metric space with $f : X \to Y$ continuous. Show that $(f(X), d_Y|_{f(X)})$ is also separable.*

1.7 The Banach fixed-point theorem

Consider a metric space (X, d) and a continuous mapping $f : X \to X$. An element $x \in X$ is called a **fixed point** of f if

$$x = f(x).$$

There are several theorems stating conditions on X and f that are sufficient for the existence of a fixed point: one of the oldest is the result below which was proved by Stefan Banach in Lwów in 1922. The Banach fixed point theorem (also known in the literature as the *contraction mapping theorem*) provides sufficient conditions for the existence *and also uniqueness* of a fixed point and, in addition, it is constructive: the fixed point can be approximated by a sequence whose convergence rate can be estimated. A standard application of Banach's theorem is found in the theory of ordinary differential equations as a main ingredient in the proof of Picard–Lindelöf's theorem about existence and uniqueness of a solution.

Theorem 1.7.1 (Banach fixed point theorem)**.** *Let (X, d) be a complete metric space and $f : X \to X$ be a contraction, cf. Definition 1.4.7. Then f has a unique fixed point. Moreover, for any arbitrary $x_0 \in X$, the sequence $(x_n)_{n\in\mathbb{N}}$ recursively defined by*

$$x_k := \begin{cases} f(x_0) & \text{if } k = 1 \\ f(x_{k-1}) & \text{if } k \geq 2 \end{cases} \tag{1.6}$$

converges to the fixed point of f.

Proof. First, if such fixed point exists then it is unique. Indeed, assume that there exists $x, y \in X$ such that $x = f(x)$ and $y = f(y)$ then

$$d(x, y) = d(f(x), f(y)) \leq Ld(x, y)$$

where L is the Lipschitz constant of f. Consequently, $(1 - L)d(x, y) \leq 0$. Since $L < 1$, this implies that $d(x, y) \leq 0$ and thus $x = y$.
Let us now prove existence. For this, let $x_0 \in X$ be arbitrary and let $(x_n)_{n\in\mathbb{N}}$ denote the sequence as defined in (1.6). By recursion we readily obtain

$$d(x_n, x_{n+1}) \leq L^n d(x_1, x_0), \quad n \in \mathbb{N}.$$

Therefore, for $p < q$,

$$\begin{aligned} d(x_p, x_q) \leq & d(x_p, x_{p+1}) + d(x_{p+1}, x_{p+2}) + ... + d(x_{q-1}, x_q) \\ & \leq (L^p + ... + L^{q-1})d(x_1, x_0) \\ & \leq \Big(\sum_{n=p}^{q-1} L^n\Big) d(x_1, x_0). \end{aligned}$$

Since $L < 1$, the series $\sum_{n\in\mathbb{N}} L^n$ converges and the last inequalities above implies that $(x_n)_{n\in\mathbb{N}}$ is a Cauchy sequence. As X is complete, $(x_n)_{n\in\mathbb{N}}$ converges to some $x \in X$. Since f is continuous, $f(x_n) \to f(x)$ by Theorem (1.4.2). From $x_{n+1} = f(x_n)$ we hence deduce $x = f(x)$. $\square$

Corollary 1.7.2. *Let (X, d) be a complete metric space and let $f : X \to X$ be a mapping. Assume that there exists $r \in \mathbb{N}^*$ such that f^r is a contraction, where the r-th iterate of f is recursively defined by*

$$f^r := \begin{cases} f & \text{if } r = 1 \\ f \circ f^{r-1} & \text{if } r \geq 2\ . \end{cases}$$

Then f has a unique fixed point.

Proof. Since X is complete and f^r is a contraction, there exists by Theorem 1.7.1 a unique $x \in X$ such that $f(x) = x$. This implies that $f^r(f(x)) = f(f^r(x)) = f(x)$. Consequently $f(x) = x$ since the fixed point of f^r is unique.
Now if there exists a another $y \in X$ such that $f(y) = y$, then $f^r(y) = y$ and thus $x = y$. $\square$

From the proof of Theorem 1.7.1 we see that this theorem can actually be generalized in the following fashion.

Theorem 1.7.3. *Let (X, d) be a complete metric space and $f : X \to X$ a mapping such that the iterate f^n satisfies*

$$d(f^n(x), f^n(y)) \leq L_n d(x, y) \qquad \text{for all } x, y \in X \text{ and all } n \in \mathbb{N} \tag{1.7}$$

where $L_n \geq 0$ such that $\sum_{n \in \mathbb{N}} L_n < \infty$. Then f has a unique fixed point.

The proof is left as an exercise.

1.7.1 Exercises

E 1.7.4. *Give the proof of Theorem 1.7.3.*

E 1.7.5. *Let $X := \mathbb{N}^* = \{1, 2, 3, \cdots\}$ and $d : X \times X \longrightarrow \mathbb{R}_+$ given by*

$$d(n, n) = 0 \text{ and } d(n, m) = 10 + \frac{1}{m} + \frac{1}{n} \text{ if } n \neq m.$$

1. *Prove that (X, d) is a complete metric space.*

2. *Let $f : X \to X$ where $f(n) = n + 1$. Prove that that for $n \neq m$*

$$d(f(m), f(n)) < d(m, n)$$

but f is not a contraction.

1.8 Compact metric spaces

In the course 1144 you have already encountered the important concept of compact subsets $S \subset \mathbb{R}^n$. According to the theorem of Heine-Borel, the class of compact subsets of $\mathbb{R}^n$ is identical with the class of closed and bounded subsets. More precisely: If S is closed and bounded and $\{O_i : i \in J\}$ is a collection of open intervals such that $S \subset \bigcup_{i \in J} O_i$, then there exists a finite subcollection $\{O_1, O_2, ..., O_n\}$ such that

$$S \subset \bigcup_{i=1}^{n} O_i \ .$$

This implies, by definition, that S is compact. On the other hand, given a subset $S \subset \mathbb{R}$ for which any open cover has a finite subcover, then S is closed and bounded. The aim of this section is to generalize the concept of compactness to arbitrary metric spaces.

1.8.1 Heine-Borel property

Let (X, d) be a metric space. An **open cover** of X is a collection of open subsets $\{O_i : i \in J\}$ of X such that $X = \bigcup_{i \in J} O_i$. A **subcover** of $\{O_i : i \in J\}$ is a collection $\{O_i : i \in I\}$ where $I \subset J$ and the subcover is finite if and only if I is finite. Finally, X is said to have the **Heine-Borel property** if every open cover of X has a finite subcover.

Definition 1.8.1. *A metric space (X, d) is said to be **compact** if every open cover of X has a finite subcover.*

The definition above does *not* state that X is compact if it has finite open cover: each metric space is open and thus is a (finite) open cover of itself. It is required that from any given open cover, we can extract a finite subcover.

Example 1.8.2. 1) *Every finite metric space is compact.*

2) *The real line $\mathbb{R}$ is not compact. In fact, we have $\mathbb{R} \subset \bigcup_{n \in \mathbb{N}}]-n, n[$. If we assume that there exists $n_1, n_2, ..., n_m \in \mathbb{N}$ such that $\mathbb{R} \subset \bigcup_{i=1,...,m \in \mathbb{N}}]-n_i, n_i[$, then we get that $\mathbb{R} \subset]-N, N[$, where $N := max\{n_1, n_2, ..., n_m\}$. This cannot be true.*

3) *Let (X, d) be a discrete metric space. If X is finite, then (X, d) is clearly compact. If however X is infinite, X is not compact since the open cover*

$$X = \bigcup_{x \in X} \{x\}$$

has no finite subcover. Accordingly, a discrete metric space (X, d) is compact if and only if X is finite.

Proposition 1.8.3. *Each compact metric space is bounded.*

Proof. Let $x_0 \in X$. Then $X = \bigcup_{n \in \mathbb{N}} B(x_0, n)$. Since X is compact there exist $n_1, n_2, ..., n_p$ such that

$$X = \bigcup_{i=1}^{p} B(x_0, n_i).$$

We set $N := max\{n_1, n_2, \ldots, n_p\}$, thus $X \subset B(x_0, N)$. Hence X is bounded. □

Remark 1.8.4. *Compactness of a metric space can also be described in terms of closed sets. Indeed, a metric space is compact if and only if for every collection of closed sets with an empty intersection one can find a finite subcollection whose intersection is also empty.*

An immediate consequence of Remark 1.8.4 is the following fact.

Proposition 1.8.5. *Let $(F_n)_{n \in \mathbb{N}}$ be a decreasing sequence of non-empty closed set of a compact metric space (X, d). Then*

$$\bigcap_{n \in \mathbb{N}} F_n \neq \emptyset.$$

Proof. Assume that $\bigcap_{n \in \mathbb{N}} F_n = \emptyset$. Since X is compact, there exists $F_{n_1}, F_{n_2}, \ldots, F_{n_p}$ such that $\bigcap_{i=1}^{p} F_{n_i} = \emptyset$. Setting $N := max\{n_1, n_2, \ldots, n_p\}$, one obtains $F_N = \bigcap_{i=1}^{p} F_{n_i} = \emptyset$. which is contradiction. □

We have seen in Proposition 1.3.11 that the conclusion of Proposition 1.8.5 is correct for complete metric space provided that $\operatorname{diam}(F_n) \to 0$. However, if this condition fails to hold and the space is not compact, the above conclusion is not true in general. For instance, for $\mathbb{R}$ we have $\bigcap_{n \in \mathbb{N}}]-\infty, n] = \emptyset$.

1.8.2 The Bolzano-Weierstrass characterization of compact metric spaces

The Bolzano-Weierstrass theorem states that *every bounded* sequence of real numbers has a cluster point, i.e., a convergent subsequence. We will prove in this section that a metric space is compact if and only if each sequence (which is not necessarily bounded!) has a cluster point.

First we need some preliminary results. A metric space is said to be **pre-compact** if for all $\varepsilon > 0$ there exists a finite open cover of X of open balls of radius $\varepsilon > 0$, i.e., for every $\varepsilon > 0$ there exists $x_0, x_1, \ldots, x_n$ in X such that

$$X = \bigcup_{i=1}^{n} B(x_i, \varepsilon).$$

Furthermore, we say that a metric space (X, d) has the **Bolzano-Weierstrass property** if every sequence in X has a cluster point.

Lemma 1.8.6. *Every metric space (X, d) with the Bolzano-Weierstrass property is pre-compact.*

Proof. We prove the lemma by contradiction. For this, assume that there exists $\varepsilon > 0$ such that X cannot be covered by a finite number of open balls of radius ε. Let $x_0 \in X$. Then $B(x_0, \varepsilon) \neq X$. Therefore, there exists $x_1 \in X$ such that

$$d(x_0, x_1) \geq \varepsilon$$

Again, since $B(x_0, \varepsilon) \cup B(x_1, \varepsilon) \neq X$, there exists $x_2 \in X$ such that $d(x_0, x_2) \geq \varepsilon$ and $d(x_1, x_2) \geq \varepsilon$. Suppose now that we have constructed vectors $x_0, x_1, \ldots, x_n$ such that $d(x_i, x_j) \geq \varepsilon$ for all $i, j = 0, 1, \ldots, n$ with $i \neq j$. Since

$$\bigcup_{i=1}^{n} B(x_i, \varepsilon) \neq X$$

there exists $x_{n+1} \in X$ such that

$$d(x_i, x_{n+1}) \geq \varepsilon \text{ for } i = 0, 1, \ldots, n.$$

Now, using pre-compactness, let $x \in X$ be the limit of a subsequence $(x_{n_k})_{k \in \mathbb{N}}$ of (x_n). Thus there exists $k_{\varepsilon/3} \in \mathbb{N}$ such that if $k \geq k_{\varepsilon/3}$

$$\varepsilon \leq d(x_{n_k}, x_{n_k+1}) \leq d(x_{n_k}, x) + d(x_{n_k+1}, x) < \frac{\varepsilon}{3} + \frac{\varepsilon}{3} = 2\frac{\varepsilon}{3}$$

which is a absurd. □

Let (X, d) be a metric space having the Bolzano-Weierstrass property and let $\{O_i : i \in J\}$ be an open cover of it. If there exists $\varepsilon > 0$ such that each open $B(x, \varepsilon)$ is contained in O_i for some $i \in J$, then X would be compact. Indeed, by Lemma 1.8.6, there exists $x_0, x_1, \ldots, x_n$ in X such that $X = \bigcup_{i=1}^{n} B(x_i, \varepsilon)$ and if each $B(x_i, \varepsilon) \subset O_i$ for some O_i, then we would have

$$X = \bigcup_{i=1}^{n} B(x_i, \varepsilon) \subset \bigcup_{i=1}^{n} O_i.$$

Let us now show that such ε does actually exist.

Lemma 1.8.7. *Let (X, d) be a metric space that has the Bolzano-Weierstrass property and let $\{O_i : i \in J\}$ be a open cover of it. Then there exists $\varepsilon_0 > 0$ such that*

$$B(x, \varepsilon_0) \subset O_i \quad \textit{for some } i_0 \in J\ .$$

Proof. Assume the assertion to be false. Then for every $\varepsilon > 0$ there exists $x \in X$ such that

$$B(x, \varepsilon) \not\subset O_i \text{ for any } i \in J.$$

In particular, for each $n \in \mathbb{N}$ and $\varepsilon = \frac{1}{n}$ there exists $x_n \in X$ such that

$$B\left(x_n, \frac{1}{n}\right) \not\subset O_i \text{ for any } i \in J. \tag{1.8}$$

Since X has the Bolzano-Weierstrass property, $(x_n)_{n\in\mathbb{N}}$ has a convergent subsequence $(x_{n_k})_{k\in\mathbb{N}}$ whose limit is denote by x. Then $x \in O_{i_0}$ for some $i_0 \in J$, because $\{O_i : \ i \in J\}$ is a cover of X. We then see that there exists $\varepsilon_0 > 0$ such that

$$B(x, \varepsilon) \subset O_{i_0} \ ,$$

as O_{i_0} is open. Take now K large enough that

$$d(x_{n_K}, x) < \frac{\varepsilon_0}{2} \quad \text{and} \quad \frac{1}{n_K} < \frac{\varepsilon_0}{2} \ .$$

Let now $y \in B(x_{n_K}, \frac{1}{n_K})$. Then $d(y, x_{n_K}) < \frac{1}{n_K}$ and thus

$$\begin{aligned} d(y, x) &\leq d(y, x_{n_K}) + d(x_{n_K}, x) \\ &\leq \frac{1}{n_K} + d(x_{n_K}, x) \leq \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon \ . \end{aligned}$$

Thus y belongs to $B(x, \varepsilon)$ and hence also to O_{i_0}. Therefore, $B(x_{n_K}, \frac{1}{n_K}) \subset O_{i_0}$, in contradiction to (1.8). □

Now we are able to prove the main result of this section.

Theorem 1.8.8 (Bolzano-Weierstrass Theorem)**.** *A metric space is compact if and only if it satisfies the Bolzano-Weierstrass property.*

Proof. If X has the Bolzano-Weierstrass property, then Lemma 1.8.6, Lemma 1.8.7 and the remark above show that X is indeed compact. For the converse we give two proofs.

First proof:Assume now that (X, d) is a compact metric space and suppose that there exists a sequence $(x_n)_{n\in\mathbb{N}} \subset X$ which has no cluster point. Then for every $x \in X$, there exists $\varepsilon_x > 0$ such that the set

$$\{n \in \mathbb{N} \mid x_n \in B(x, \varepsilon_x)\}$$

is finite (see Remark 1.3.16). On the other hand, $\{B(x, \varepsilon_x), \ x \in X\}$ is an open cover of X and by compactness there exists $y_0, y_1, \ldots, y_p$ such that

$$X = \bigcup_{i=1}^{p} B(y_i, \varepsilon_{y_i}).$$

This however yields a contradiction since at least one y_j would be a cluster point of $(x_n)_{n\in\mathbb{N}}$.

Second proof: Let $(x_n)_{n\in\mathbb{N}}$ be a sequence on X and let

$$F_p := \overline{\{x_n, \ n \geq p\}}, \ p \in \mathbb{N}$$

Thus (F_p) is a decreasing sequence of non-empty closed sets. Thus $\bigcap_{p\in\mathbb{N}} F_p \neq \emptyset$. But by Proposition 1.3.15 $\bigcap_{p\in\mathbb{N}} F_p$ coincides with the set of cluster points of $(x_n)_{n\in\mathbb{N}}$. □

Theorem 1.8.9. *Each compact metric space is complete and separable.*

Proof. The first statement follows from Theorem 1.8.8 and Proposition 1.3.17. Now from Lemma 1.8.6 it follows that, for every $n \in \mathbb{N}$, there exists $x_0^n, x_1^n, \dots, x_{p_n}^n$ in X such that

$$X = \bigcup_{i=0}^{p} B\big(x_{n_i}, \frac{1}{n}\big).$$

Thus $D := \{x_j^n, n \in \mathbb{N} \text{ and } j = 0, 1, \dots, p_n\}$ is dense in X. □

Using the induced metric, we can also define compactness for subsets of metric spaces. In fact, we will say that $K \subset X$ is **compact** if the metric subspace (K, d_K) is compact. More precisely, $K \subset X$ is compact if and only if each sequence $(x_n) \subset K$ has a subsequence which converges in K. Equivalently, using Remark 1.2.18 one can easily see that $K \subset X$ is compact if and only if each open cover of A with open sets of X has a finite subcover. Let us mention a few further elementary but useful properties of compact metric spaces and sets.

Proposition 1.8.10. *Let $(x_n)_{n\in\mathbb{N}}$ be a convergent sequence in a metric space (X, d) and let x denotes its limit. Then the set*

$$A := \{x_n : \ n \in \mathbb{N}\} \cup \{x\}$$

is compact.

Proof. Let $\{O_i, \ i \in J\}$ be an open cover of A. Since $x \in A$, there exists $i_0 \in J$ such that $x \in O_{i_0}$. Now because $x_n \to x$, there exists $n_0 \in \mathbb{N}$ such that for all $n \geq n_0, x_n \in O_{i_0}$. For $n \leq n_0$, there $i_n \in J$ such that $x_n \in O_{i_n}$. Then one has $A \subset \cup_{i\in I} O_i$, where $I := \{i_n : \ n \leq n_0\} \cup \{i_0\}$. Thus the claim is proved since I is finite. □

Remark 1.8.11. *Note that the conclusion of Proposition 1.8.10 is no longer true when the limit of the sequence is missing (i.e. when $A = \{x_n : \ n \in \mathbb{N}\}$). Take for instance*

$$A := \left\{\frac{1}{n} : \ n \in \mathbb{N} \setminus \{0\}\right\}$$

and let

$$O_1 = \left(\frac{1}{2}, 2\right) \text{ and } O_n = \left(\frac{1}{n+1}, \frac{1}{n+1}\right) \text{ for } n \geq 2.$$

Then $\{O_n : n \in \mathbb{N}\}$ is an open cover of A where each O_n contains exactly one element of A which is $\frac{1}{n}$. Consequently, A cannot be compact.

Proposition 1.8.12. *Let (X, d) be a metric space. Then the following assertions hold.*

1) If X is compact and $F \subset X$ is closed, then F is compact.

2) If F is a compact subset of X, then F is closed and bounded.

Proof. 1) Obvious.

2) F is closed, since it contains the cluster point of its sequence and thus the limits of its convergent sequences. The boundedness was proven in Proposition 1.8.3. □

Remark 1.8.13. *As we have already mentioned, the converse of the second statement of Proposition 1.8.12 holds for $X = \mathbb{R}^n$, but not in general! For example, if (X, d) is an infinite discrete metric space, then we have seen in Example 1.8.2 that $A \subset X$ can only be compact if it is finite.*

1.8.3 Continuous functions in compact metric space

Proposition 1.8.14. *Let (X, d_X) and (Y, d_Y) be two metric spaces and $f : X \to Y$ be a continuous function. Then, for each compact subset $K \subset X$ the image $f(K)$ is compact and f is uniformly continuous from K to Y.*

Proof. Let $(y_n)_{n\in\mathbb{N}} \subset f(K)$. Then there exists $(x_n)_{n\in\mathbb{N}} \subset Y$ with $x_n = f(y_n)$. Since K is compact, there exists a subsequence $x_{n_k} \to x \in K$. Since f is continuous, this implies that $y_{n_k} = f(x_{n_k}) \to f(x) \in f(X)$. Thus $f(K)$ is compact.
Now assume by contradiction that $f : K \to Y$ is not uniformly continuous. Then there exists $\varepsilon > 0$ and two sequences $(x_n)_{n\in\mathbb{N}}$ and $(y_n)_{n\in\mathbb{N}}$ in K such that

$$d_X(x_n, y_n) \le \frac{1}{n} \text{ and } d_Y(f(x_n), f(y_n)) \ge \varepsilon.$$

Since K is compact, $(x_n)_{n\in\mathbb{N}}$ and $(y_n)_{n\in\mathbb{N}}$ have convergent subsequences $(x_{n_k})_{k\in\mathbb{N}}$ and $(y_{n_k})_{k\in\mathbb{N}}$ converging to the same point $x \in K$. By the continuity of f there exists $k_0 \in \mathbb{N}$ such that for all $k \ge k_0$ one has

$$\varepsilon < d_Y(f(x_{n_k}), f(y_{n_k})) \le d_Y(f(x_{n_k}), f(x)) + d_Y(f(y_{n_k}), f(x)) \le \frac{\varepsilon}{2}$$

which is absurd. □

Remark 1.8.15. *A consequence of Proposition 1.8.14, is that the image $f(F)$ of a closed subset F of is closed in Y provided that X is compact. This is not true in general as we have seen in Remark 1.4.6.*

Remark 1.8.16. *Let (X, d_X) and (Y, d_Y) be two metric spaces such that X is compact. Let $f : X \to Y$ be a continuous function. Proposition 1.8.14 shows that for given $y \in Y$ a solution of the equation*

$$f(x) = y \tag{1.9}$$

can be obtained by approximation. More precisely, if y can be approximated by a sequence $(y_n)_{n\in\mathbb{N}} \subset Y$ (i.e., $y_n \to y$ is Y) and if we assume that there exists a sequence $(x_n)_{n\in\mathbb{N}} \subset X$ of solutions of

$$f(x_n) = y_n,$$

then we can extract a subsequence $(x_{n_k})_{k\in\mathbb{N}}$ of $(x_n)_{n\in\mathbb{N}}$ that converges to a solution of (1.9).

A function f between two metric spaces (X, d_X) and (Y, d_Y) is a **homeomorphism** if f is continuous, bijective and f^{-1} is continuous. We say that f is an **isometry** from (X, d_X) to (Y, d_Y) if

$$d_Y(f(x), f(y)) = d_X(x, y) \quad \text{for all } x, y \in X.$$

Proposition 1.8.17. *Let (X, d_X) and (Y, d_Y) be two metric spaces and let f be an isometry from X to Y. Then the following assertions hold.*

1) f is injective.

2) If (X, d_X) is complete, then the image $f(X)$ is complete in Y.

3) If f is surjective, then f is a homeomorphism from X to Y.

In general , a continuous and bijective function need not have a continuous inverse. Consider for example the identity function id : $(\mathbb{R}, d') \to (\mathbb{R}, d)$, where d' is the discrete metric and d is the usual metric. Then this function is is bijective and continuous but its inverse (clearly, id : $(\mathbb{R}, d) \to (\mathbb{R}, d')$) is not continuous. Things look better under a compactness assumption.

Proposition 1.8.18. *Let (X, d_X) and (Y, d_Y) be two metric spaces and $f : X \to Y$ be a continuous and bijective function. If X is compact, then f is a homeomorphism.*

Proof. From Proposition 1.4.4 it suffices to show that the image by f of each closed set of X is closed in Y. Let $F \subset X$ be closed. Since X is compact, F is also compact. By Proposition 1.8.14, $f(F)$ is compact and then closed in Y. $\square$

If A a compact set of $\mathbb{R}$, then $\inf(A)$ and $\sup(A)$ belong to A. In fact, A is in particular bounded and thus $\sup(A), \inf(A) < \infty$. There exists two sequences $(x_n)_{n\in\mathbb{N}}$ and $(y_n)_{n\in\mathbb{N}}$ in A such that $x_n \to \inf(A)$ and $y_n \to \sup(A)$. Since A is closed we conclude that $\inf(A)$ and $\sup(A) \in A$.

From this remark and Proposition 1.8.14 we deduce that each continuous real value function on a compact metric space is bounded and attains a maximum and a minimum on X.

Theorem 1.8.19 (Extreme value theorem)**.** *Let (X, d) be a compact metric space and $f : X \to \mathbb{R}$ be a continuous function. Then f attains a maximum and a minimum value, i.e., there exist $a, b \in X$ such that*

$$f(a) = \inf_{x\in A} f(x) \ \ \textit{and} \ \ f(b) = \sup_{x\in A} f(x).$$

1.8.4 Exercises

E 1.8.20. *Give a proof of Remark 1.8.4.*

E 1.8.21. *Suppose that (X, d) is a compact metric space and $\{K_n\}$ a decreasing sequence of closed subsets of X. Setting $K := \bigcap_n K_n$, show that* $\operatorname{diam}(K_n) \to \operatorname{diam}(K)$.

1.9 Notes and suggestions for further learning

One does not even need distances in order to define an abstract notion of open set! An axiomatic theory of open and closed sets (and therefore of continuity and convergence) has been developed since the 1910s by Felix Hausdorff in Greifswald and many other subsequent mathematicians, leading to the birth of the mathematical field of *Topology*: we refer the interested reader to the course 1354.
In general, a family of subsets $\mathcal{T}$ of a non-empty set X is called **topology** if the following holds:

(O1) $X \in \mathcal{T}$ and $\emptyset \in \mathcal{T}$,

(O2) if $U_\alpha : \alpha \in J$ is any collection of elements of $\mathcal{T}$, then $\bigcup_{\alpha\in J} U_\alpha \in \mathcal{T}$

(O3) if, for any $n \in \mathbb{N}$, $U_1, U_2, , \cdots , U_n \in \mathcal{T}$, then $\bigcap_{i=1}^n U_i \in \mathcal{T}$.

The pair $(X, \mathcal{T})$ is called a topological space and the elements of $\mathcal{T}$ are called *open sets* (with respect to $\mathcal{T}$). Therefore, what we have proved in Theorem 1.2.5 is that the notion of openness introduced in Definition 1.2.3 is in fact consistent with the topological definition of openness.

Not only topological results, but even topological *notions* are useful in analysis and well beyond. Let us mention a nice application to linear algebra. You have already seen in the course 1143 that symmetric matrices (or hermitian matrices in the complex case) are diagonalizable, i.e., they are similar to a diagonal matrices. However, non-symmetric matrices seem to be "many more" than symmetric ones, so what should be done if a non-symmetric matrix is given? Well. it turns out that we cannot be too far away from a diagonalizable matrix anyway. Indeed, one can prove that diagonalizable $n \times n$ matrices with complex values are dense in set of all $n \times n$ matrices with respect to the topology induced by entrywise convergence. While the proof is based on more or less elementary linear algebra, it is the idea of using denseness to qualitatively suggest how "common" objects are which is rather topological indeed.

1.10 Solutions

Solution 1 (E 1.4.10)**.** *Assume that $f : X \to Y$ is continuous at $x \in X$ and let U be a neighbourhood of $f(x)$. Then is $V := f^{-1}(U)$ is a neighbourhood of x and $f(V) \subset U$. Let us show the converse assertion. Let U be a neighbourhood of $f(x)$. Then there exists by assumption a neighbourhood of V of x such that $f(V) \subset U$. Since*

$$V \subset f^{-1}(f(V)) \subset f^{-1}(U)$$

then is $f^{-1}(U)$ also a neighbourhood of x. Therefore $f : X \to Y$ is continuous at $x \in X$.

Solution 2 (E 1.1.4)**.** *First, we start from condition $c)$ in Definition 1.1.1 to obtain*

$$d(x, z) - d(y, z) \leq d(x, y) \ .$$

Interchanging $x \leftrightarrow y$ and using symmetry we obtain

$$d(y, z) - d(x, z) \leq d(x, y) \ .$$

From this, (1.1) follows. Next, using (1.1) *we obtain that*

$$\begin{aligned} |d(x, y) - d(u, y)| &\leq d(x, u) \ , \\ |d(v, u) - d(y, u)| &\leq d(y, v) \ . \end{aligned}$$

Adding both inequalities and using triangle inequality then yields the result.

Solution 3 (E 1.1.5)**.** *We check conditions $a) - c)$ of Definition 1.1.1. Condition $b)$ is clear and $a)$ follows from the fact that the map* $\arctan : \mathbb{R} \cup \{\pm\infty\} \to [-\frac{\pi}{2}, +\frac{\pi}{2}]$ *is bijective. Finally, condition $c)$ follows from the triangle inequality.*

Solution 4 (E 1.1.6)**.** *We check conditions $a) - c)$ of Definition 1.1.1.*

i) *If $d_X(x, y) = 0$, then $x_n = y_n$ for all $n \in \mathbb{N}$. On the other hand, $d_X(x, x) = 0$ for all $x \in X$.*

ii) *Is clear.*

iii) *Let $x, y, z \in X$ be given. Using the triangle inequality we obtain*

$$|x_n - z_n| \leq |x_n - y_n| + |y_n - z_n|, \quad \forall n \in \mathbb{N}.$$

Hence,

$$\begin{aligned} \sup_{n \in \mathbb{N}} |x_n - z_n| &\leq \sup_{n \in \mathbb{N}} \big(|x_n - y_n| + |y_n - z_n|\big), \\ &\leq \sup_{n \in \mathbb{N}} |x_n - y_n| + \sup_{n \in \mathbb{N}} |y_n - z_n|. \end{aligned}$$

We conclude that (X, d_X) is indeed a metric space.

Solution 5 (E 1.1.7)**.** *(X, d_X) is not a metric space. To see this, choose $x, y \in X$ such that $x_n \neq y_n$ for infinitely many n to obtain $d_X(x, y) = \infty$.*

Solution 6 (E 1.1.8)**.** *$i)$ The map $d_1 : (x,y) \mapsto |x^2 - y^2|$ is not a metric on $\mathbb{R}$, since condition $a)$ of Definition 1.1.1 is not satisfied. In fact, for instance $d_1(-2,2) = 0$.*
$ii)$ Clearly d_2 satisfies conditions $a) - b)$ of Definition 1.1.1. For given $x, y, z \in \mathbb{R}$ we have

$$|x^3 - z^3| \leq |x^3 - y^3 + y^3 - z^3| \leq |x^3 - y^3| + |y^3 - z^3|.$$

Thus d_2 is a metric on $\mathbb{R}$.
$iii)$ The mapping $(x,y) \mapsto e^{\frac{1}{|x-y|}}$ is never equal 0 and thus condition $a)$ of Definition 1.1.1 is not satisfied. Therefore d_3 is not a metric on R.

Solution 7 (E 1.1.9)**.**

1. *We first note that, since f, g are continuous, the integral and hence $d_1(f,g)$ is well defined as a Riemannian integral. Also, symmetry of the metric is clear and from $d_1(f,f) = 0$ we readily conclude $f = 0$. To see this note that $|f|$ is positive and continuous. Finally, using the triangle inequality and linearity of the integral yields the result.*

2. *Since any continuous function on a compact interval is bounded we readily conclude that $d_\infty(f,g)$ is well defined. Also, symmetry is obvious and $d_\infty(f,f) = 0$ clearly implies $f = 0$. Using triangle inequality then yields the result.*

Solution 8 (E 1.2.19)**.** *Consider $A = \{x\}$ where $x \in X$ is any point. In order to show that A is closed we show that $X \setminus A$ is open. Therefore, let $y \in X \setminus A$ be any point and consider the ball $B_r(y)$ with $r < d(x,y) < \infty$. Then $B_r(y) \subset X \setminus A$ and we conclude that $X \setminus A$ is indeed open.*

Solution 9 (E 1.2.20)**.**

a) *Assume that A is dense and $X \setminus A \neq \emptyset$. We then choose some element $y \in X \setminus A$. Since A is dense we have that $y \in \overline{A}$ and by the definition of the closure we conclude that, for any ball $B_r(y)$, one has $B_r(y) \cap A \neq \emptyset$. For r small enough one has, by the definition of the discrete metric, that $B_r(y) = \{y\}$. This, however, yields a contradiction.*

b) *" $\Rightarrow$ ": Let A be dense in X and $B_r(x)$ some open ball with center $x \in X$. Since A is dense we immediately conclude that $A \cap B_r(x) \neq \emptyset$. " $\Leftarrow$ ": One has $\overline{A} := \{x \in X \mid B_r(x) \cap A \neq \emptyset,\ \forall r > 0\}$. Hence, $\overline{A} = X$.*

Solution 10 (E 1.2.21)**.** *The assertion "$i)$" is obvious. Assume that $d(x,A) = 0$. Then for each $r > 0$ we have $d(x,A) < r$. Thus for each $r > 0$ there exists $y \in A$ such that $d(x,y) < r$. So for each $r > 0$ we have $B(x,r) \cap A \neq \emptyset$. This means by Definition that 1.2.9 $x \in \overline{A}$. Thus "$ii)$" holds. As conclusion we obtain the following characterization*

$$\overline{A} = \{x \in X : d(x,A) = 0\}.$$

In fact, the inclusion $\overline{A} \supset \{x \in X : d(x,A) = 0\}$ follows from (ii). On the other hand, if $x \in \overline{A}$ then $i)$ implies that $d(x,A) \leq d(x,\overline{A}) = 0$ and thus $\overline{A} \subset \{x \in X : d(x,A) = 0\}$.

Solution 11 (E 1.2.22)**.** *Let $x, y \in A \cup B$. If x, y belong to A (respectively to B) then $d(x,y) \leq$ $\mathrm{diam}(A)$ (respectively $d(x,y) \leq \mathrm{diam}(B)$). Now if $x \in A$ and $y \in B$ then for arbitrary $a \in A$ and $b \in B$ the triangle inequality implies*

$$d(x,y) \leq d(x,a) + d(a,b) + d(b,y) \leq \mathrm{diam}(A) + d(a,b) + \mathrm{diam}(B).$$

Thus

$$\begin{aligned} d(x,y) &\leq \mathrm{diam}(A) + \mathrm{diam}(B) + \inf_{a \in A} \inf_{b \in B} d(a,b) \\ &= \mathrm{diam}(A) + \mathrm{diam}(B) + d(A,B). \end{aligned}$$

Solution 12 (E 1.2.23). (i) *Since* $A \subset \overline{A}$ *then* $\mathrm{diam}\ (A) \leq \mathrm{diam}\ (\overline{A})$. *Let* $x, y \in \overline{A}$. *Then for each* $\varepsilon > 0$ *there exist* $x', y' \in A$ *such that*

$$d(x, x') \leq \varepsilon/2 \ \text{ and } \ d(y, y') \leq \varepsilon/2.$$

Then

$$\begin{aligned} d(x, y) &\leq d(x, x') + d(x', y') + d(y, y') \\ &\leq \varepsilon + d(x', y') \\ &\leq \varepsilon + \mathrm{diam}\ (A). \end{aligned}$$

It follows that

$$\sup_{x,y \in \overline{A}} d(x, y) \leq \mathrm{diam}\ (A),$$

i.e., $\mathrm{diam}(\overline{A}) \leq \mathrm{diam}'(A)$. *We conclude that indeed* $\mathrm{diam}(\overline{A}) = \mathrm{diam}(A)$.
(ii) *If* $x \in A \cap B$, *then* $d(A, B) \leq d(x, x) = 0$. *Thus* $d(A, B) = 0$.
(iii) *Set* $\alpha := \mathrm{diam}(A) + d(A, B) + \mathrm{diam}(B)$. *Let* $x, y \in A \cup B$:

1. *if* $x, y \in A$ *or* $x, y \in B$ *then* $d(x, y) \leq \mathrm{diam}\ \ A \leq \alpha$,
2. *if* $x \in A$ *and* $y \in B$ *then* $d(x, y) \leq d(A, B) \leq \alpha$.

We conclude that $\mathrm{diam}\ (A \cup B) \leq \alpha$.

Solution 13 (E 1.2.24).

1. *Let* $x, y, z \in X$. *Assume for example that* $d(x, y) < d(y, z)$. *Since* d *is an ultrametric we have*

$$d(y, z) \leq \max\{d(y, x), d(x, z)\}.$$

If we had $d(x, z) < \max\{d(x, y), d(y, z)\} = d(y, z)$, *then*

$$d(y, z) \leq \max\{d(y, x), d(x, z)\} < \max\{d(y, z), d(y, z)\}$$

which is impossible.

2. *Let* $x \in X$ *and* $r > 0$. *Let* $y \in \overline{B(x, r)}$. *There exists* $\varepsilon > 0$ *and* $z \in X$ *such that* $d(x, z) < r$ *and* $d(y, z) < \varepsilon$. *In particular for* $\varepsilon = r$ *we obtain that* $d(x, z) < r$ *and* $d(y, z) < r$. *Then*

$$d(y, x) \leq \max\{d(x, z), d(y, z)\} < r,$$

so $y \in B(x, r)$. *It follows that* $B(x, r) = \overline{B(x, r)}$. *Let now* $y \in B(x, r)$, *if* $z \in B(y, r)$ *then*

$$d(x, z) \leq \max\{d(x, y), d(y, z)\} < r.$$

This implies the inclusion $B(y, r) \subset B(x, r)$. *The* $B(x, r) \subset B(y, r)$ *follows by a similar argument.*

3. *Let* $x \in X$ *and* $r > 0$. *Let* $y \in \overline{B}(x, r)$. *If* $d(x, y) \leq r$, *then* $y \in B(x, r) \subset \overline{B}(x, r)$ *and we have* $y \in B(y, r) \subset \overline{B}(x, r)$ *if* $d(x, y) = r$. *Thus* $\overline{B}(x, r)$ *is a neighborhood of every of its elements and thus open. The second statement can be proved as in* (2).

Solution 14 (E 1.3.18). *We have to show that each convergent sequence* (x_n) *has a unique limit* x. *Suppose that* $x_n \to x$ *and also* $x_n \to y$ *with* $x \neq y$. *By definition we have* $d(x_n, x) \leq \frac{\varepsilon}{2}$ *and* $d(x_n, y) \leq \frac{\varepsilon}{2}$ *for* n *large enough. Hence*

$$d(x, y) \leq d(x_n, y) + d(x_n, x) \leq \varepsilon\ .$$

Since ε *can be arbitrarily small, we conclude that* $d(x, y) = 0$ *and hence* $x = y$ *which is a contradiction.*

Solution 15 (E 1.3.19)**.** *Let $x_0 \in \overline{\{x\}}$. By Theorem 1.3.5 there exists a sequence $(x_n) \subset \{x\}$ such that $x_n \to x$ as $n \to \infty$. Since $x_n \in \{x\}$ for every $n \in \mathbb{N}$ it follows that $x_n = x$ for every $n \in \mathbb{N}$. Thus $x_0 = \lim_{n\to\infty} x_n = x$ and then $\overline{\{x\}} = \{x\}$. The result then follows from Proposition 1.2.11.*

Solution 16 (E 1.3.20)**.** *Assume that $(x_n)_{n\in\mathbb{N}} \subset X$ is a Cauchy sequence. Then for each $\varepsilon > 0$ there exists $n_0 \in \mathbb{N}$ such that for all $p, q \geq n \geq n_0$*

$$d(x_p, x_q) \leq \varepsilon.$$

Thus for each $\varepsilon > 0$ there exists $n_0 \in \mathbb{N}$ such that for all $p, q \geq n_0$

$$\operatorname{diam}(A_n) = \sup_{p,q\geq n_0} d(x_p, x_q) \leq \varepsilon.$$

i.e., $\operatorname{diam}(A_n) \longrightarrow 0$. *Conversely, if* $\operatorname{diam}(A_n) \longrightarrow 0$ *then for each $\varepsilon > 0$ there exists $n_0 \in \mathbb{N}$ such that* $\operatorname{diam} A_n = \sup_{p,q\geq n_0} d(x_p, x_q) \leq \varepsilon$ *for all $n \geq n_0$. In particular for $n = n_0$ we have $d(x_p, x_q) \leq \varepsilon$ for all $p, q \geq n_0$. Thus $(x_n)_{n\in\mathbb{N}}$ is a Cauchy sequence.*

Solution 17 (E 1.3.21)**.** *Whenever X consists of only one element, i.e. $X = \{x\}$, then every sequence is the constant sequence and hence convergent. On the other hand, suppose that every bounded sequence converges in X. Let $x \neq y$ be two different elements and consider the sequence (z_n) with $z_{2n} = x$ and $z_{2n+1} = y$. This sequence is obviously bounded and hence, by assumption, it converges to some limit z. However, every subsequence of (z_n) must hence also converge to z which implies that $z = y = x$ which is a contradiction.*

Solution 18 (E 1.3.22)**.**

(a) Let $(x_n)_{n\in\mathbb{N}}$ be a Cauchy sequence in $\ell^1(\mathbb{N})$, i.e., $d(x_n, x_m) \leq \varepsilon$ for all $n, m \geq n_0$. This means that

$$\sum_i |x_n(i) - x_m(i)| \leq \varepsilon$$

where $x_n(i)$ denotes the i-th component of $x_n \in \ell^1(\mathbb{N})$. Hence, for each fixed value i, $|x_n(i) - x_m(i)|$ is a Cauchy sequence in the reals and hence converges. We define this limit to be x with $x_i = \lim_{n\to\infty} x_n(i)$. The goal is now to show that $x \in \ell^1$ and $x_n \to x$. For this, consider

$$\sum_{i=N}^{M} |x_i| = \lim_{m\to\infty} \sum_{i=N}^{M} |x_m(i)| \leq \varepsilon,$$

for $N, M \geq N_0$. Hence we have $x \in \ell^1(\mathbb{N})$. Also,

$$\sum_{i=0}^{\infty} |x_i - x_n(i)| = \sum_{i=0}^{M} |x_i - x_n(i)| + \sum_{i=M+1}^{\infty} |x_i| + \sum_{i=M+1}^{\infty} |x_n(i)|.$$

Now fix $n \geq n_0$ and then choose M so large that the two latter terms are each smaller than $\frac{\varepsilon}{3}$. Note that the last term is then smaller than $\frac{\varepsilon}{3}$ for all $m \geq n$. Hence, for fixed M, we can then choose n so large that also the first term becomes smaller than $\frac{\varepsilon}{3}$. This shows that $d(x_n, x) \to 0$ which concludes the proof.

(b) In order to show that $\overline{c_{00}} = \ell^1$, we prove that for every $x \in \ell^1$ and $\varepsilon > 0$, there exists $y \in c_{00}$ such that $d(y, x) \leq \varepsilon$. For a given $\varepsilon > 0$ choose N large enough such that

$$\sum_{n=N}^{\infty} |x_n| \leq \varepsilon.$$

Defining $y_n := x_n$ for $n \leq N$ and $y_n = 0$ for $n > N$ we have $y \in c_{00}$ and $d(y, x) \leq \varepsilon$. This is the desired result.

Solution 19 (E 1.4.11). *Let $O \subset Y$ be an open set. Since the metric considered on X is the discrete metric every subset of X is open. This implies that $f^{-1}(O)$ is also open. Hence f is continuous by Proposition 1.4.4.*

Solution 20 (E 1.4.12). *Let $x_0 \in \mathbb{R}$ and $\varepsilon > 0$. Denote by $\delta' > 0$ the non-negative root of the polynomial $\eta^2 + 2|x_0|\eta - \varepsilon$. Thus for all $y \in \mathbb{R}$ with $|x_0 - y| \leq \eta'$ we have*

$$\begin{aligned} |f(x_0) - f(y)| &= |x_0^2 - y^2| \\ &\leq |(x_0 - y)^2 + 2x_0(x_0 - y)| \\ &\leq |(x_0 - y)^2| + 2|x_0||(x_0 - y)| \leq \eta'^2 + 2|x_0|\eta' = \varepsilon. \end{aligned}$$

Thus f is continuous. Take now $\varepsilon = 1$ and for each $\eta > 0$ take $x = \frac{-\eta}{4} - \frac{2}{\eta}$ and $y = x + \frac{\eta}{2}$. Then $|x - y| = \frac{\eta}{2} < \eta$ but $|f(x) - f(y)| = |x^2 - y^2| = 2 > \varepsilon = 1$. We conclude that f is not uniformly continuous.

Solution 21 (E 1.4.13). *Let $(x_n)_{n\in\mathbb{N}}$ be a Cauchy sequence in (X, d_X). From the uniform continuity of f we conclude that $(f(x_n))$ forms a Cauchy sequence in (Y, d_Y) and due to completeness there exists $y \in Y$ such that $f(x_n) \to y$ as $n \to \infty$. Since f is bijective there exists $x \in X$ with $f(x) = y$. Hence we have $f(x_n) \to f(x)$. Finally, due to the continuity of f^{-1} we conclude that $x_n \to x$ which completes the proof.*

Solution 22 (E 1.6.5). *Let $A \subset X$ be dense in (X, d_X) with $A := \{a_n\}$. Consider any open ball $B_\varepsilon(y_0)$ with $y_0 \in Y$ and $\varepsilon > 0$. Then*

$$f^{-1}\big(B_\varepsilon(y_0) \cap f(A)\big) = f^{-1}\big(B_\varepsilon(y_0)\big) \cap A.$$

Since f is continuous we conclude that $f^{-1}\big(B_\varepsilon(y_0)\big)$ is open in X. Furthermore, since A is dense, we see that $f^{-1}\big(B_\varepsilon(y_0)\big) \cap A \neq \emptyset$ which implies that $B_\varepsilon(y_0) \cap f(A) \neq \emptyset$ and hence that $f(A)$ is dense in $(f(X), d_Y|_{f(X)})$.

Solution 23 (E 1.7.4). *To prove uniqueness, assume that $x \neq y$ are both fixpoints. Then*

$$d(x, y) = d(f^n(x), f^n(y)) \leq L_n d(x, y)\ . \tag{1.10}$$

Since $\sum_{n\in\mathbb{N}} L_n$ converges we conclude that $L_n \to 0$ and hence we obtain $d(x, y) = 0$. Now, for $x_0 \in X$ we consider here the sequence

$$x_n := f^n(x_0),\ n \in \mathbb{N}.$$

Then for all $n \in \mathbb{N}$ we have $x_{n+1} = f(x_n)$ and

$$\begin{aligned} d(x_n, x_{n+1}) &= d(f^n(x_0), f^{n+1}(x_0)) \\ &= d(f^n(x_0), f^n(f(x_0))) \leq L_n d(x_0, f(x_0)). \end{aligned}$$

Thus, for $p < q$,

$$\begin{aligned} d(x_p, x_q) &\leq d(x_p, x_{p+1}) + d(x_{p+1}, x_{p+2}) + ... + d(x_{q-1}, x_q) \\ &\leq (L_p + ... + L_{q-1}) d(x_0, f(x_0)) \\ &\leq (\sum_{n=p}^{q-1} L_n) d(x_0, f(x_0)). \end{aligned}$$

Thus $(x_n)_{n\in\mathbb{N}}$ is a Cauchy sequence in X and thus converges to some $x \in X$. On the other hand, (1.10) implies in particular that f is continuous and hence

$$f(x) = \lim_{n\to\infty} f(x_n) = \lim_{n\to\infty} x_{n+1} = x.$$

Solution 24 (E 1.7.5)**.** 1) *It is easy to check that (X, d) is a metric space. Let $(x_n) \subset X$ be a Cauchy sequence. Then for all $\varepsilon > 0$ there exists $n_0 \in \mathbb{N}$ such that $d(x_p, x_q) = 10 + \frac{1}{x_p} + \frac{1}{x_q} \le \varepsilon$ for all $p, q \ge n_0$. In particular for $\varepsilon < 10$, this implies that $x_p = x_q$ for all $p, q \ge n_0$. We conclude that (X, d) is complete.*
2) *For all $n \neq m$*

$$d(f(m), f(n)) = 10 + \frac{1}{m+1} + \frac{1}{n+1} < 10 + \frac{1}{m} + \frac{1}{m} = d(n, m).$$

However, f is not contractive, otherwise f would have a fixed point by Theorem 1.7.1 which is impossible.

Solution 25 (E 1.8.20)**.** *Assume that (X, d) is compact and let $\{A_i\}$ be a collection of closed sets such that $\bigcap_i A_i = \emptyset$. We have*

$$X \setminus \bigcap_i A_i = \bigcup_i (X \setminus A_i) = X.$$

Since X is compact there exists a finite subcover, i.e.,

$$X = \bigcup_i^N (X \setminus A_i) = X \setminus \bigcap_i^N A_i.$$

Hence $\bigcap_i^N A_i = \emptyset$.
Conversely, let $\{O_i\}$ be a open cover of X, i.e., $X = \bigcup_i O_i$. Then $\{B_i := X \setminus O_i\}$ is a collection of closed sets such that $\bigcap_i B_i = \emptyset$. Hence there exists a finite subcollection $\{B_i\}_n^N$ with $\bigcap_i^N B_i = \emptyset$. This implies that $X = \bigcup_i^N O_i$ and hence X is compact.

Solution 26 (E 1.8.21)**.** *Since X is compact, it is bounded (according to Proposition 1.8.3). Hence the sequence $\operatorname{diam}(K_n)$ is bounded from above and below. Since it is decreasing we conclude that $\lim_{n \longrightarrow \infty} \operatorname{diam}(K_n)$ exists. Also, by construction,*

$$\lim_{n \longrightarrow \infty} \operatorname{diam}(K_n) \ge \operatorname{diam}(K).$$

Now, since X is compact we conclude that each K_n is a compact set. Furthermore, since the metric d is a continuous function we conclude that, for any K_n, there exist $x_n, y_n \in K_n$ such that

$$d(x_n, y_n) = \operatorname{diam}(K_n).$$

Due to compactness, each sequence (x_n), (y_n) has a cluster point. Restricting ourselves to a subsequence we can assume that $x_n \to x$ and $y_n \to y$. We readily observe that $x, y \in K$. Hence,

$$diam K \ge d(x, y) = \lim_n d(x_n, y_n) = \lim_n \operatorname{diam}(K_n).$$

Therefore, $\operatorname{diam}(K) = \lim_n \operatorname{diam}(K_n)$.