Diskrete Mathematik und Optimierung

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Technical Report feu-dmo020.10
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2000 Mathematics Subject Classification: 05B35, 05C05, 05C57
Keywords: maker-breaker game, tree pair, unique single element exchange, wheel, basis exchange

On a base exchange game on graphs

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April 6, 2010

Abstract

We consider the following maker-breaker game on a graph G that has a partition of the edge set E into two spanning trees E_1 and E_2. Initially the edges of E_1 are purple and the edges of E_2 blue. Maker and breaker move alternately. In a move of the maker a blue edge is coloured purple. The breaker then has to recolour a different edge blue in such a way that the purple and the blue edges are spanning trees again. The goal of the maker is to exchange all colours, i.e. to make E_1 blue and E_2 purple. We prove that a sufficient but not necessary condition for the breaker to win is that the graph contains an induced K_4. Furthermore we characterize the structure of a partition of a wheel into two spanning trees and show that the maker wins on wheels W_n with $n \geq 4$ and provide an example of a graph where, for some partitions, the maker wins, for some others, the breaker wins.

Key words: maker-breaker game, tree pair, unique single element exchange, wheel, basis exchange
MSC: 05B35, 05C05, 05C57, 91A43

1 Introduction

A graph $G = (V, E)$ is a *tree pair* if its edge set admits a partition $E = E_1 \dot\cup E_2$ into two spanning trees, i.e. such that (V, E_1) and (V, E_2) are trees.

Consider the following game which is played by two players, a *maker* Alice and a *breaker* Bob, on a tree pair $G = (V, E)$ given with a partition $E = E_1 \dot\cup E_2$ of the edge set into two spanning trees. During the game, some edges are in the dynamic set P of *purple edges*, the other edges in the dynamic set B of *blue edges*. Initially, $P = E_1$ and $B = E_2$. The players move alternately, the maker begins. A move of the maker consists in colouring a blue edge e purple, i.e. $P \longrightarrow P \cup \{e\}$ and $B \longrightarrow B \setminus \{e\}$. After that the breaker must colour a purple edge $f \neq e$ blue in such a way that the purple and blue edges each form a spanning tree again. If the maker can enforce that the purple and blue edges are completely exchanged in a finite number of steps, i.e. $P = E_2$ and $B = E_1$, the maker wins. Otherwise, i.e. if the breaker can achieve an infinite sequence of moves without happening the winning configuration for Alice, the breaker wins.

This paper deals with the question: Given a tree pair $G = (V, E)$ and a partition of the edge set $E = E_1 \dot\cup E_2$ into a purple and a blue spanning tree, which player has a winning strategy for the game described above?

We call this game *base exchange game for graphs*. This is motivated by the following generalizations of the game for matroids proposed by White [5], see also [4].

A *base pair geometry* is a matroid M on a ground set E such that $E = X_1 \dot\cup X_2$ for two bases X_1, X_2 of M. White defines the following maker-breaker game on a base pair geometry with bases X_1 and X_2: the maker chooses $a \in X_1$ and the breaker must choose $b \in X_2$ such that $X_1 - a + b$ and $X_2 - b + a$ are new bases for the next move. If after a finite series of moves X_1 and X_2 are exchanged, the maker wins. We call this game $W(1)$. In another game, which we call $W(2)$, the maker is allowed to choose $a \in X_1$ or $b \in X_2$. The breaker then must recreate two new bases different from the bases of the previous move.

Our game is the special case of $W(1)$ for graphic matroids. Note that $W(2)$ is the same for graphic and cographic matroids. Our game has more strict rules than $W(2)$. The difference can be seen by the example of the K_4 (see Section 3). Here the breaker has a winning strategy for our game, but not for $W(2)$. White conjectures that for every regular matroid the maker has a winning strategy for $W(2)$.

Neil White's motivation to study the games comes from the research on the following exchange properties of matroids. In [4] he considers matroids M with ground set S and compatible pairs $X = (X_1, X_2)$ and $Y = (Y_1, Y_2)$ of bases of M. *Compatibility* means that each element of S is contained in the same number of X_is as of Y_js. We say that the pair $X' = (X_1 - a + b, X_2 - b + a)$ is obtained by a *unique exchange* from the pair $X = (X_1, X_2)$ if $a \in X_1$ and there is only exactly one $b \in X_2$ such that $X_1 - a + b$ and $X_2 - b + a$ are bases of M. If a base pair Y can be obtained from a base pair X by a series of unique exchanges, we write, following White, $X \simeq_1 Y$. If a $Y = (Y_1, Y_2)$ can be obtained from $X = (X_1, X_2)$ by a series of unique exchanges and permutation of the X_is, we write $X \simeq_2 Y$. White defines $\mathrm{UE}(\mathbf{1})'$ resp. $\mathrm{UE}(\mathbf{2})'$ as the class of matroids such that for all compatible basis pairs X and Y, $X \simeq_1 Y$ resp. $X \simeq_2 Y$. White conjectures that the class of regular matroids is contained in $\mathrm{UE}(\mathbf{2})'$.

This conjecture is very much related to the conjecture that for regular matroids the maker has a winning strategy in game $W(2)$. Note that the class of matroids on which the maker has a winning strategy in game $W(2)$ contains $\mathrm{UE}(\mathbf{2})'$, since if the maker chooses $b \in X_2$ this can be seen as permuting X_1 and X_2, choosing b from the now first entry of the base pair and repermuting.

Obviously, the class of matroids on which the maker has a winning strategy in game $W(1)$ contains $\mathrm{UE}(\mathbf{1})'$. By a result of White $\mathrm{UE}(\mathbf{1})'$ is the class of series-parallel matroids. We will show that the class of matroids where the maker wins is richer, namely it contains all wheels except K_4.

The paper is organized as follows. In Section 2 we introduce basic terminology and results for the base exchange game on graphs. We prove that the breaker wins on tree pairs that contain an induced K_4 in Section 3. However, in Section 4 we show that a breaker-win graph does not necessarily contain an induced K_4. Section 5 deals with the structure of partitions of the edge set of wheels into spanning trees. These results are needed for Section 6 in which we show that the maker has a winning strategy on wheels that are not the K_4. In Section 7 we give an example of a tree pair in which the maker wins for some partitions, and the breaker for other. We assume familiarity with graph theory, basic knowledge of matroid theory and use standard notation. Standard references are [1, 3].

2 General results on the base exchange game

For the discussion of the game we introduce an auxiliary digraph, the so-called *graph of forced transformations* G^F. Its vertices are all pairs (E_1, E_2) of disjoint spanning trees with $E = E_1 \dot\cup E_2$ of the base graph $G = (V, E)$. We have an arc $((E_1, E_2), (E_1', E_2'))$ if and only if in the game there is an edge $e \in E_2$ such that, if it is coloured by the maker, there is only a single edge $f \in E_1$ the breaker may colour as feasible answer in such a way that $E_1' = (E_1 \cup \{e\}) \setminus \{f\}$ and $E_2' = (E_2 \setminus \{e\}) \cup \{f\}$.

The following obvious Proposition is the basis for our further analysis:

Proposition 1. *If G^F is strongly connected, then the maker has a winning strategy for the base exchange game on the graph G for any starting partition into two spanning trees.*

The following proposition that the game is well-defined is the special case of the well-known symmetric base exchange property of matroid theory.

Proposition 2. *Let $G = (V, E)$ be a tree pair with a partition $E = P \dot\cup B$ of the edge set into two spanning trees. Then*

$$\forall b \in B \exists p \in P : (P \setminus \{p\} \cup \{b\}, B \setminus \{b\} \cup \{p\}) \text{ is a partition into trees.}$$

Proof. Let $C(P, b)$ denote the fundamental circuit of P and b and $C^*(B, b)$ the cut induced by the two components of $B \setminus \{b\}$. Then $|C(P, b) \cap C^*(B, b)|$ is even, the intersection contains b and

$$\forall p \in C(P, b) \cap C^*(B, b) : (P \setminus \{p\} \cup \{b\}, B \setminus \{b\} \cup \{p\})$$

is a partition into trees. □

3 Graphs where the breaker wins

The main purpose of this section is to show that the breaker has a winning strategy if a tree pair contains the complete graph K_4 as an induced subgraph.

Lemma 3. *(a)* K_4^F *has three components.*

(b) If (P, B) *is an ordered partition of the edge set of* K_4 *into two trees, then* (P, B) *and* (B, P) *lie in different components of* K_4^F*.*

(c) If the maker plays the unique non-forced move the breaker has a feasible move that does not leave the component.

Proof. (a) Any partition of K_4 into two trees consists of two P_4s, i.e. paths on four vertices. Let *abcd* denote such a P_4. Then the move of the breaker is forced if and only if the maker does not close a C_4, i.e. plays edge *ad*. The resulting configurations of forced moves are listed in Fig. 1. Hence the component of the purple *abcd* consists of purple $\{abcd, abdc, bacd, badc\}$ and hence of four ordered partitions. As the number of P_4s in K_4 is $\frac{1}{2}4! = 12$ the claim follows by symmetry.

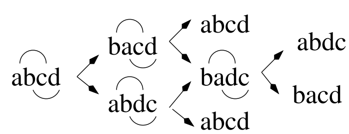

Figure 1: A component of K_4^F

(b) The above analysis yields that the purple *bdac* lies in a different component than *abcd*.

(c) By symmetry, again, it suffices to consider the case that the starting configuration is a purple P_4 *abcd* and the maker plays *ad*. Now the breaker recolours *bc* to purple which yields a purple *badc* in the same component. □

Summarizing the last Lemma implies:

Theorem 4. *The breaker has a winning strategy on the* K_4 *for any starting configuration.*

Theorem 5. *If a tree pair* G *contains a tree pair* H *as an induced subgraph, and the breaker has a winning strategy for* H*, then the breaker has a winning strategy for* G*.*

Proof. If a blue edge outside H is recoloured to purple by the maker, then recolouring a purple edge in H would mean that the blue graph in H has two edges more than the purple graph in H, therefore there is a blue cycle. So the answer on recolouring outside H must also be an edge outside H. Therefore the breaker can use his strategy on H, and if the maker plays in the complement, the breaker plays in the complement. In this way the spanning trees of H cannot be exchanged and thus the same holds for the global spanning trees. □

Corollary 6. *For a tree pair that contains a K_4 as induced subgraph the breaker has a winning strategy.*

4 A breaker-win graph without induced K_4

By D_6 we denote the 2-sum of two K_4s "without glueing edge" (see Fig. 2). Clearly, K_4 is a minor but not an induced subgraph of D_6.

Theorem 7. *For any partition $E = E_1 \cup E_2$ of the edge set of D_6 into two trees, the breaker has a winning strategy.*

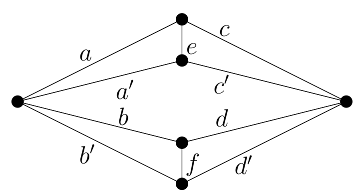

Figure 2: The graph D_6

Proof. We consider two cases.

Case 1: E_1 contains an edge of each of the pairs $\{a, a'\}, \{b, b'\}, \{c, c'\}, \{d, d'\}$ such that these four edges do not form a path. By symmetry, we may assume, that $A := \{a, b, c', d'\} \subseteq E_1$. Note, that any further edge $x \in E \setminus A$ will complement A to a tree such that $E \setminus (A \cup \{x\})$ is a tree as well. Therefore, if $x = E_1 \setminus A$ and the maker recolours $y \in E \setminus (A \cup \{x\})$ to purple, the breaker recolours x to blue. Hence, A will never change its colour and the breaker wins.

Case 2: First we will show that otherwise E_1 must be disjoint from one of the pairs $\{a, a'\}, \{b, b'\}, \{c, c'\}, \{d, d'\}$. Assume not, then E_1 must contain one edge of each pair, which alltogether form a path. We may assume, by symmetry, that $\{b, a, c, d'\} \subseteq E_1$ implying $\{f, d, b'\} \in E_2$. As E_2 forms a tree at least one of a', c' must be in E_1, hence E_1 contains an edge from each pair (namely either $\{a, b, c', d'\}$ or $\{a', b, c, d'\}$) which alltogether do not form a path.

Hence we may assume that $\{a, a'\} \subseteq E_2$. Therefore, E_1 must contain an induced P_4 in the lower half, w.l.o.g. $\{b', d, f\} \subseteq E_1$, $\{b', d, d'\} \subseteq E_1$ or $\{b, b', d\} \subseteq E_1$ and, again by symmetry, we may assume that E_1 is one from $\{e, c, b', d, f\}$, $\{e, c, b', d, d'\}$ or $\{e, c, b, b', d\}$ (see Fig. 3).

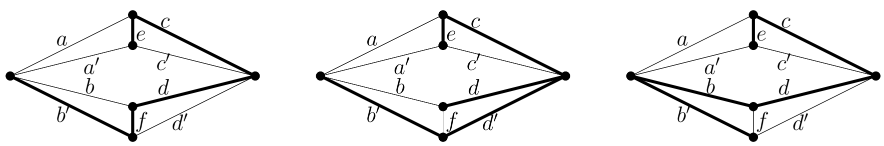

Figure 3: The three possible trees E_1 (fat edges) in Case 2

We will show that the breaker can assure, that $\{e, d, b'\}$ never change their colour. The maker's moves and the answers of the breaker are listed in Table 1.

$E_1 := \{e, c, b', d, f\}$				
a	a'	b	c'	d'
c	c	f	c	f

$E_1 := \{e, c, b', d, d'\}$				
a	a'	b	c'	f
c	c	d'	c	d'

$E_1 := \{e, c, b, b', d\}$				
a	a'	c'	d'	f
c	c	c	b	b

Table 1: Fixing b', d, e.

In any case the breaker reinstalls a partition where E_1 is disjoint from either $\{a, a'\}$ or $\{c, c'\}$ and which contains the vertical edge adjacent to this pair, i.e. e, and two independent edges from the other side, namely b', d. Hence, by symmetry, the breaker can ensure that $\{e, d, b'\}$ never change their colour and wins.

□

Remark 8. *It can be shown [2] that the graph of forced transformations D_6^F has exactly 8 components, four of size 6 corresponding to Case 1 of the above proof and four of size 12 corresponding to Case 2.*

5 On the structure and the number of partitions of a wheel into two trees

We start with a crucial of observation.

Proposition 9. *Let $W_n = (V, E)$ denote the n-wheel and $E = E_1 \dot\cup E_2$ be a partition of the edges into two trees. Let $S \subseteq E$ denote the spokes and $S_1 := S \cap E_1$, $R \subseteq E$ the rim edges and $R_1 := R \cap E_1$. Let c denote the hub and $v_0, \ldots, v_{n-1}$ the outer vertices of W_n and $S_1 = \{cv_{i_1}, cv_{i_2}, \ldots, cv_{i_k}\}$ in cyclic clockwise order. Then*

$$\begin{aligned} R_1 &= R \setminus \{v_{i_1}v_{i_1+1}, v_{i_2}v_{i_2+1}, \ldots, v_{i_k}v_{i_k+1}\} \text{ or} \\ R_1 &= R \setminus \{v_{i_1}v_{i_1-1}, v_{i_2}v_{i_2-1}, \ldots, v_{i_k}v_{i_k-1}\} \end{aligned}$$

where indices are taken modulo n.

Proof. Since $|V| = n + 1$, and E_1 is a set of edges of a spanning tree we must have $|E_1| = n$ and hence $|R_1| = n - k$. If e is a rim edge adjacent to two spokes from E_2 it must be in E_1, since E_2 has no triangle. Hence, each element from $R \setminus R_1$ is of the form $v_{i_j}v_{i_j+1}$ or $v_{i_j}v_{i_j-1}$. Assume that there exists $v_{i_j}v_{i_j+1}$ as well as $v_{i_\ell}v_{i_\ell-1}$ in E_2 and $cv_{i_j+1}, cv_{i_\ell-1} \in E_2$. If $j = \ell$, E_2 would contain the cycle $v_{i_j+1}v_{i_j}, v_{i_j}v_{i_j-1}, v_{i_j-1}c, cv_{i_j+1}$, thus necessarily $j \neq \ell$. We may choose j, ℓ such that cv_{i_j} precedes cv_{i_ℓ} in S_1. But this contradicts the fact that E_1 induces a connected graph. □

Proposition 10. *Let $W_n = (V, E)$ denote the n wheel and $E = E_1\dot{\cup}E_2$ a partition of the edges. Let S, R, S_1, R_1 be as in Proposition 9 and*

$$\begin{aligned} R_1 &= R \setminus \{v_{i_1}v_{i_1+1}, v_{i_2}v_{i_2+1}, \ldots, v_{i_k}v_{i_k+1}\} \text{ or} \\ R_1 &= R \setminus \{v_{i_1}v_{i_1-1}, v_{i_2}v_{i_2-1}, \ldots, v_{i_k}v_{i_k-1}\} \end{aligned}$$

where indices are taken modulo n.

If $\emptyset \neq S_1 \neq S$, then E_1 and E_2 both induce trees.

Proof. First note that if in R_1 the left rim edge is missing at each spoke, the same holds for R_2, vice versa. The same holds if the right rim edge is missing. Hence it suffices to show that E_1 induces a tree. Since $|E_1| = n$ this follows if E_1 is acyclic. The latter is clear, since in each path between two consecutive spokes exactly one edge is missing. The claim follows. □

Theorem 11. *The number of partitions of the edge set of the wheel W_n into two trees is $2^n - 2$.*

Proof. By Propositions 9 and 10 there is a bijection between the oriented proper subsets of S and the trees whose complements are trees as well. We have $2 \cdot (2^n - 2)$ oriented proper subsets of S, and we have counted each partition twice. The claim follows. □

Corollary 12. *The number of partitions of the element set of the n-whirl into two bases is $2^n - 1$.*

Proof. Compared to the wheel we have the additional partition into the spokes and the rim. □

6 The strategy of the maker for wheels

In this section we discover an important class of maker-win graphs, namely the class of wheels. Wheels are the simplest, most natural example for tree pairs with a high degree of symmetry.

Theorem 13. *Let $W_n = (V, E)$ be a wheel with $n \geq 4$ and let $E = E_1\dot{\cup}E_2$ be a partition of the edges into two spanning trees. Let the edges of E_1 be purple and those of E_2 be blue. Then the maker has a strategy in the base exchange game to force an exchange of the colours of E_1 and E_2.*

Proof. We will prove, using the following three lemmata, that W_n^F is strongly connected for $n \geq 4$. Then the theorem follows by Proposition 1. □

First we need some definitions. We use the notation from Proposition 9. By this proposition, either the purple rim edges follow the purple spokes counter-clockwise or clockwise. In the first case we speak of a *left orientation*, see Fig. 4 left, in the second of a *right orientation*, see Fig. 4 center. Every purple spoke s_{i_j} that is adjacent to a purple rim edge is called *ending spoke*. There are some special configurations. If a left orientation has only one purple spoke s_i, the configuration is called s_i*-left path*, see Fig. 4 right. Its complement (i.e. the configuration with only one blue spoke s_i) is called s_i*-left star*. In both cases, s_i is also called *special spoke*. We use analog notions for right orientations.

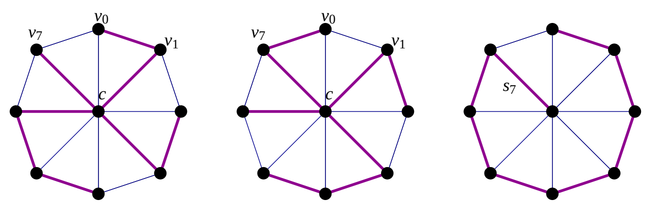

Figure 4: Left and right orientation and a left path

Lemma 14. *Any left orientation of the wheel W_n, $n \geq 4$ can be transformed into an s_i-left path, where s_i is one of the ending spokes of the left orientation and stays in the same colour.*

Proof. We proceed by induction on the number $k \geq 1$ of purple spokes. If $k = 1$ there is nothing to prove. Otherwise, we may assume $i = n - 1$. Let $s_j \neq s_{n-1}$, $j < n - 1$ be the purple spoke such that s_ℓ is blue for $j < \ell < n - 1$. The maker recolours the formerly blue rim edge $v_j v_{j+1}$ purple, creating a purple cycle $cv_j v_{j+1} \ldots v_{n-1}c$. By Proposition 9 and as $n \geq 4$, the only possibility for the breaker to reinstall a tree-pair is to recolour s_j to blue and the claim follows by induction. □

Lemma 15. *The s_i-left path of the wheel W_n, $n \geq 4$, can be transformed into the s_i-left star.*

Proof. First the maker recolours the rim edge $v_i v_{i+1}$ to purple, so the breaker is forced to make the rim edge $v_{i-1} v_i$ blue, turning the s_i-left path into the s_i-right path, see Fig. 5 left. Then the maker recolours the spoke s_{i+2} to purple, so that the breaker is forced to colour the rim edge $v_{i+1} v_{i+2}$ blue, see Fig. 5 center left. Now the maker inductively recolours the spokes s_{i+2+j}, $j = 1, 2, \ldots, n - 3$ (indices mod n), each move forcing the breaker to recolour the rim edge $v_{i+2+j-1} v_{i+2+j}$ to blue, see Fig. 5 center right. Now, we are left with the s_{i+1}-right star. In order to turn this into the s_i-left star it suffices to recolour s_{i+1} to purple, which forces the breaker to make s_i blue, see Fig. 5 right. Note that the first and last pair of moves again requires $n \geq 4$. In case $n = 3$, s_{i+1} and s_{i-1} would be neighboured, and the breakers move is not forced any more. □

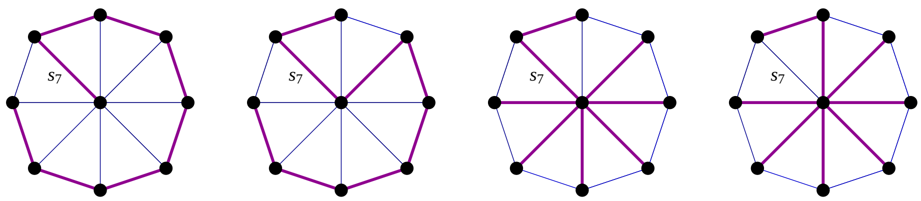

Figure 5: Transforming the s_7-left path into the s_7-left star

Lemma 16. *If C^1, C^2 are two s_l-left orientations of the wheel W_n, $n \geq 4$ with $S_1^2 \subseteq S_1^1$, where S_1^k denotes the set of purple spokes of C^k, then C^1 can be transformed into C^2.*

Proof. We proceed by induction on the number $k = |S_1^1| - |S_1^2|$, the case $k = 0$ being trivial. Let $s_i \in S_1^1 \setminus S_1^2$. The maker recolours $s_{i+1}s_i$, making it purple. Since C^1 has at least 2 purple spokes, we may choose $s_j \in S_1^1$ such that $s_m \notin S_1^1$ for $i < m < j$. In order to destroy the cycle $cs_is_{i+1}\dots s_jc$ and to reinstall a tree pair, by Proposition 9 the breaker is forced to colour s_i blue. Now, the claim follows by induction. □

Theorem 17. *W_n^F is strongly connected for $n \geq 4$.*

Proof. The following chain of arguments is depicted in Figure 6. By Lemma 14 we can transform any left orientation C^1 with spokes S_1^1 where $i \in S_1$ and $j \notin S_1$ for given $i \neq j$ into the i-left path. By Lemma 15 we can transform the i-left path into the i-left star and by Lemma 16 from this we reach any left orientation C^2 with spokes S_1^2 and $i \notin S_1^2$, $m \in S_1^2$ for any $m \neq i$. Interchanging the roles of the indices we conclude that we can transform this into C^1 and hence the subdigraph of W_n^F induced by the left orientations is stronly connected.

In the proof of Lemma 15 we, furthermore, transformed the i-left path into the $(i+1)$-right star and this into the i-left star. Since by, symmetry, the right orientations induce a strongly connected digraph as well, the claim follows.

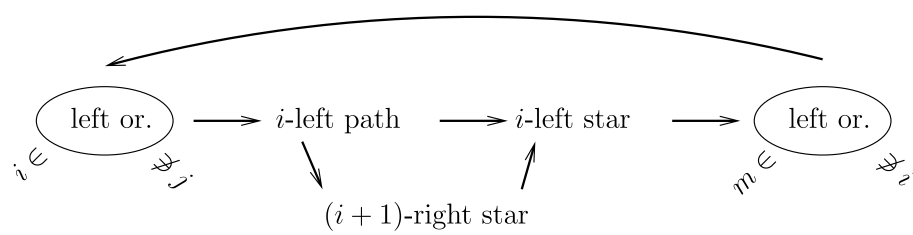

Figure 6: W_n^F is strongly connected

□

7 A non-homogeneous graph

In this section we will consider the graph $G = K_{3,3} + e$ where e is an additional edge in one of the bipartitions. We will see that for some partitions of G into two trees the breaker has a winning strategy, for others the breaker has a winning strategy. Moreover, we will see that G^F decomposes into two components, one containing 48 tree pairs, the other 24 tree pairs. The maker wins exactly on half of the tree pairs of the bigger component.

In order to be able to describe this phenomenon more in detail, we start by identifying the types of tree pairs which can occur in G. In Fig. 7 three types and their complements are depicted.

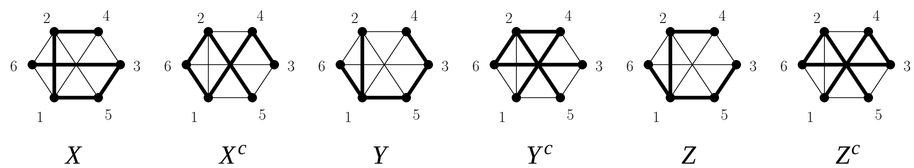

Figure 7: Types X and X^c, Y and Y^c, Z and Z^c

We say two partitions are of the same type if there is an automorphism of G transforming one into the other. It is easy to verify that in each case there are exactly 12 pairs of the same type (and 12 pairs of the complement of these types) since the autorphism group of G is $S_2 \times S_3$, where S_i denotes the permutation group on i elements. In pairs of the type X, Y, and Z the special edge $\{1, 2\}$ is purple, in the complements it is blue. Note that in a pair of type X the purple and the blue edges form a P_6, in a pair of type Z the purple edges form a generalized star $S_{1,2,2}$ and the blue edges form a P_6, and in a pair of type Y the purple edges form another generalized star $S_{1,1,3}$ whereas the blue edges form an $S_{1,2,2}$.

Theorem 18. *(a) G^F consists of two components A and B, where A contains every partition of type X^c, Y^c, Z, and Z^c and B contains those of types X and Y.*

(b) The maker wins if the starting partition is of type Z or Z^c.

(c) The breaker wins if the starting partition is of type X, X^c, Y, or Y^c.

We will prove this theorem by a series of lemmata.

Lemma 19. *The tree pairs of type X and type Y form a component of G^F. Moreover, the breaker has a strategy never to leave this component if the game is started here.*

Proof. In Fig. 8 we depict all possible results of a pair of moves, starting from X (upper row) resp. from Y (lower row). Alice recolours some edge and in most cases Bob's response is forced (grey edge). In the three non-forced moves we show Bob's possible moves in grey. In all three non-forced moves, if the breaker plays the lower edge $\{1, 5\}$, either a partition of type X or of type Y is created. In the forced moves it can be seen that also only types X or Y are created, they are denoted as X resp. Y with the permutations corresponding to the automorphisms..

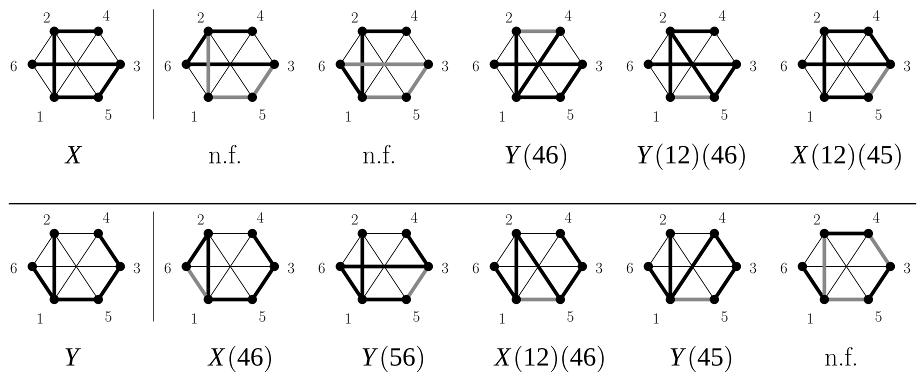

Figure 8: Moves starting with Y

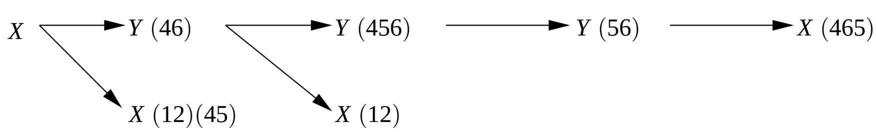

Figure 9: Paths of moves

In Fig. 9 we see paths of moves from X to $Y(4,6)$ and $X(12)(45)$, $X(12)$ resp. $X(465)$. Since $\{(12)(45), (12), (465)\}$ is a generating set of the automorphism group of G, the partitions of types X and Y form a single component of G^F. $\square$

Lemma 20. *Types X^c, Y^c, Z and Z^c are in the same component. In particular, each type Z can reach each type Z^c.*

Proof. In Fig. 10 we depict all forced and non-forced (n.f.) moves starting from the tree pair X^c, Y^c, Z, Z^c, respectively. The answer of the breaker in forced moves is the grey edge. It can be seen that there are no forced moves which obtain a partition of type X or Y.

Furthermore each permutation of each of the four partitions X^c, Y^c, Z, and Z^c can be reached from any one of them, as is proven by the paths in Fig. 11.

This means that type X^c, Y^c, Z, and Z^c form a component of G^F. $\square$

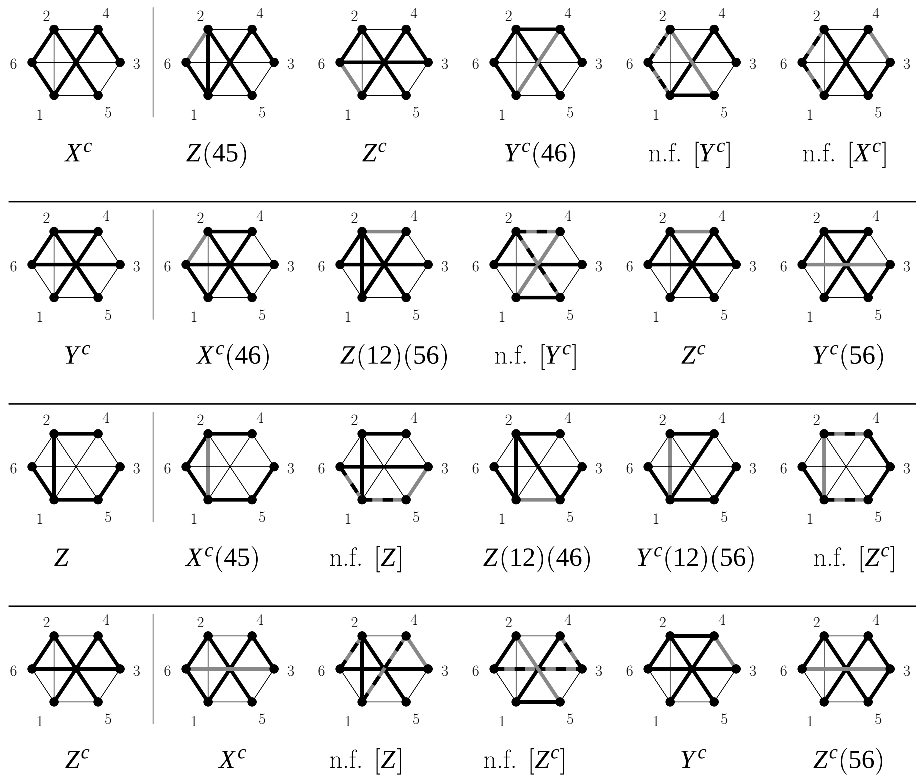

Figure 10: Moves starting with X^c, Y^c, Z, resp. Z^c

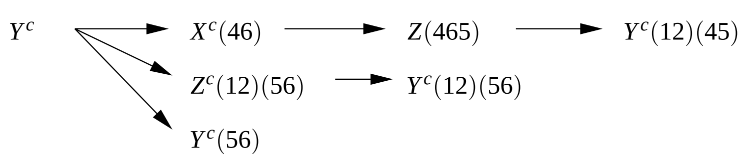

Figure 11: Paths of forced moves

Lemma 21. *If the initial partition is of type X^c or Y^c, in the non-forced moves the breaker has a strategy to obtain partitions of types X^c, Y^c, Z or Z^c again (to stay in the same component).*

Proof. In the non-forced moves of Fig. 10, if the breaker chooses the grey edge for recolouring (not the grey-black dashed edges), he also obtains a partition of a type which is displayed in brackets. This type is neither X nor Y in any case. □

This completes the proof of Theorem 18.

8 Concluding remarks

We have seen in the last section that there is a tree pair with partitions $E = E_1 \dot\cup E_2$ and $E = F_1 \dot\cup F_2$ into spanning trees such that the maker wins when the initial partition is (E_1, E_2) but the breaker wins when the initial partition is (F_1, F_2). However, the following problem is still open.

Problem 22. *Is there a tree pair $G = (V, E)$ with partition $E = E_1 \dot\cup E_2$ into spanning trees such that the maker wins when the initial partition is (E_1, E_2) but the breaker wins when the initial partition is (E_2, E_1)?*

In all our examples, if the maker has a winning strategy for a tree pair G with initial partition (E_1, E_2), the partition (E_2, E_1) was in the same component. Note that a positive answer to Problem 22 implies a positive answer to the following

Problem 23. *Is there a tree pair $G = (V, E)$ with partition $E = E_1 \dot\cup E_2$ into spanning trees such that the maker wins when the initial partition is (E_1, E_2), but (E_2, E_1) and (E_1, E_2) lie in distinct components of G^F?*

References

[1] R. Diestel: Graph Theory. Springer, Berlin 2000

[2] M. Merkel: Das Basentauschspiel für Graphen, bachelor's thesis, FernUniversität Hagen, 2009

[3] J. G. Oxley: Matroid Theory. Oxford University Press, Oxford 1992

[4] N. White: A unique exchange property for bases. Linear Algebra and its applications 31 (1980), 81–91

[5] N. White: personal communication