Diskrete Mathematik und Optimierung

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Oriented Matroids - From Matroids and Digraphs to Polyhedral Theory

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Oriented Matroids - From Matroids and Digraphs to Polyhedral Theory

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These notes are intended for participants of the MAA Shortcourse on Matroid Theory January 2011 in New Orleans. Therefore our intention is not to give an introduction into the theory of oriented matroids from scratch (as in [9]), but to recapture how they arise from matroids. Therefore, we assume basic knowledge of matroid theory.

For a gentle introduction into the theory of oriented matroids we recommend [1], the standard reference is [2].

1 Directed Planar Graphs and their Duals

1.1 Introduction

A *graph* $G = (V, E)$, where V is a finite set (of *vertices*) and $E \subseteq \binom{V}{2} \cup V$ is a finite set of *edges* (one- or two-element subsets of the vertices), may be considered as a symmetric, binary relation. If we drop the symmetry requirement we arrive at digraphs.

So, the difference between graphs and digraphs is that the arcs have an orientation from one end vertex to the other. The purpose of this section is to give an idea how we can save at least some of the orientation information to matroids, where we do no longer have vertices.

A main concern should be that the orientation is somewhat compatible with duality. Thus maybe we should start with a planar graph and its dual.

1.2 An example

Consider the following orientation of the dodekahedron (see Figure 1). How to choose the direction for the dual arcs? Here we have chosen the orientation such that the dual arc has the right-of-way, i.e. the primal arc points from the left to the right.

Figure 3 illustrates that directed circuits give rise to directed cuts and vice versa.

It seems that the orientation of the graph can be encoded as partitions of the circuits and partitions of the cuts into forward and backward arcs. If we

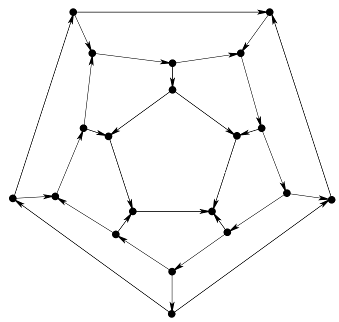


Figure 1: An orientation of the dodekahedron

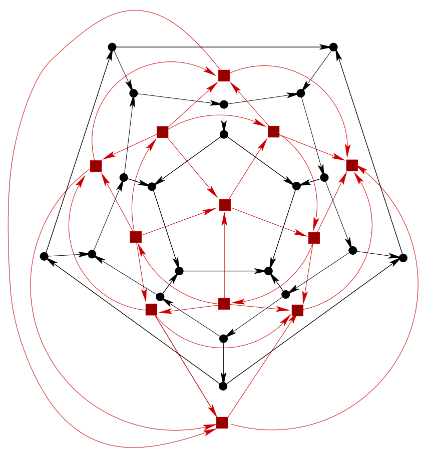


Figure 2: The oriented dual of the orientation of the dodekahedron

consider the intersection between an oriented circuit and an oriented cut in Figure 4 we observe that the patterns $\{++, --\}$ occur the same time as the patterns $\{+-, -+\}$.

This is not a property of planar graphs but is easily seen to hold for arbitrary directed graphs (see Figure 5).

2 Minty's Orientability – Regular Matroids

Let M be a matroid on a finite set E, let $\mathcal{C}$ denote its set of circuits and $\mathcal{D}$ denote its set of cocircuits. Then in [6] George J. Minty called M *orientable* if there is an (ordered) partition of all circuits $C \in \mathcal{C}$ into (C^+, C^-) and of all

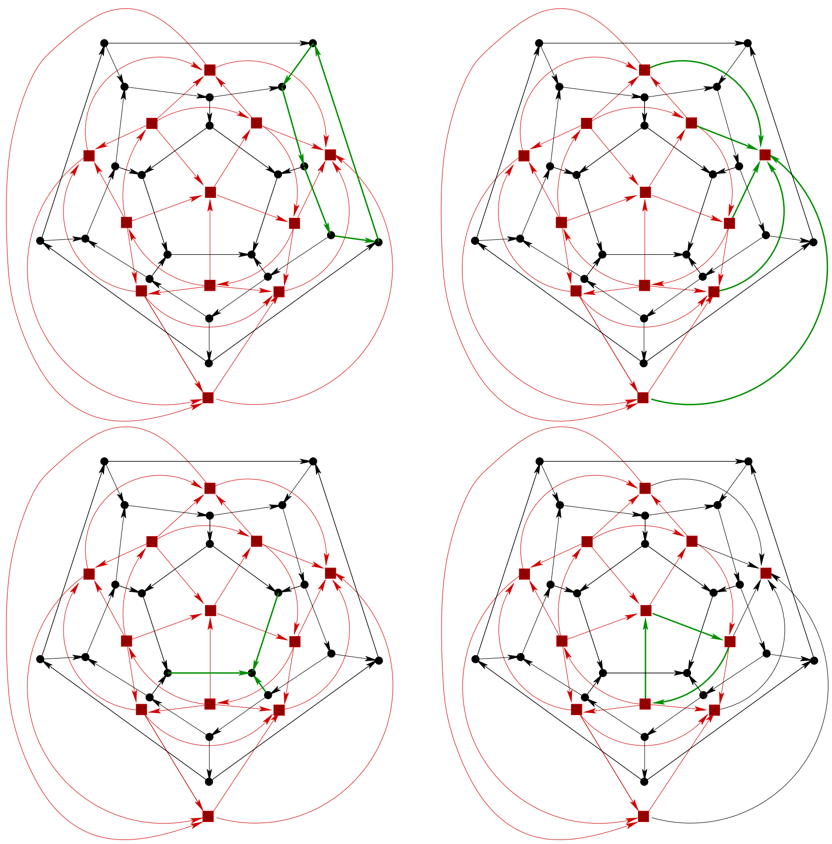


Figure 3: Duality of directed circuits and directed cuts

cocircuits $D \in \mathcal{D}$ into (D^+, D^-) such that

$$\forall C \in \mathcal{C}\, \forall D \in \mathcal{D} : |C^+ \cap D^+| + |C^- \cap D^-| = |C^+ \cap D^-| + |C^- \cap D^+|. \qquad (1)$$

We call a pair of vectors satisfying (1) *orthogonal*. Note that the order of the partition does not really matter for (1) to hold. We will frequently denote an ordered partition $\vec{C} = (C^+, C^-)$ of a set C by its signed characteristic function:

$$\chi_{\vec{C}}(e) = \begin{cases} 1 & \text{if} \quad e \in C^+ \\ -1 & \text{if} \quad e \in C^- \\ 0 & \text{otherwise.} \end{cases}$$

We will also write just $+$ and $-$ instead of 1 and -1 when considering the signed characteristic function.

Example 1. *a). If M is the* polygon matroid *or* circuit matroid *of a graph $G = (V, E)$ then choosing some orientation of the edges we find partitions of the circuits and cocircuits which show that M is orientable.*

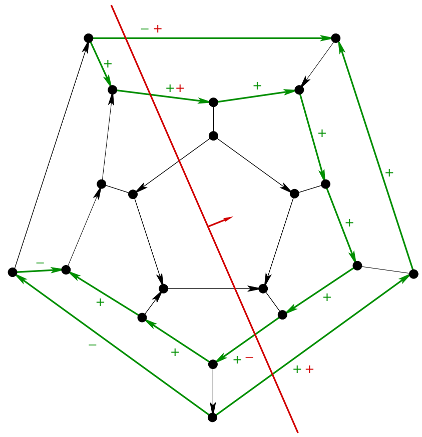


Figure 4: Intersection of an oriented cut and an oriented circuit

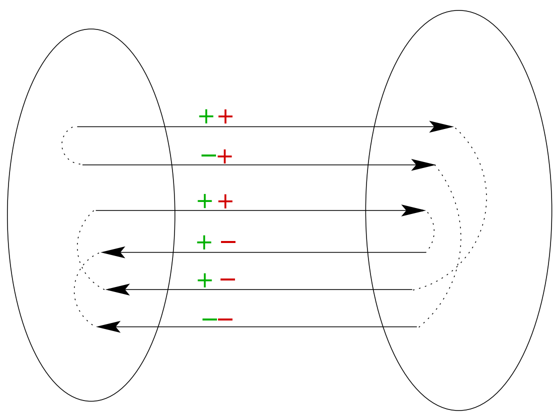


Figure 5: Intersection of an oriented cut and an oriented circuit

b). The bond matroid *of a graph is orientable, since orientability is invariant under duality.*

c). Consider the matroid R_{10} defined over $GF(2)$ by the matrix

$$\left(\begin{array}{ccccc|ccccc} 1 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 1 & 1 \end{array}\right).$$

The circuits are the non-zero vectors of minimal support in the kernel and the cocircuits the non-zero vectors of minimal support in the row space. Matrix A depicts a partition of a spanning set of the cocircuit space. We *get a partition of the circuits by considering the signs of the vectors in the kernel. Matrix A^* is a representation of the dual. The rows indicate partitions of some of the circuits of R_{10}.*

$$A = \left(\begin{array}{ccccc|ccccc} 1 & 0 & 0 & 0 & 0 & -1 & 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 & 1 & -1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 1 & -1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & -1 & 1 \\ 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 1 & -1 \end{array}\right)$$

$$A^* = \left(\begin{array}{ccccc|ccccc} 1 & -1 & 0 & 0 & -1 & 1 & 0 & 0 & 0 & 0 \\ -1 & 1 & -1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & -1 & 1 & -1 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 1 & -1 & 0 & 0 & 0 & 1 & 0 \\ -1 & 0 & 0 & -1 & 1 & 0 & 0 & 0 & 0 & 1 \end{array}\right).$$

For example, the first two rows of A^*, $(1,-1,0,0,-1,1,0,0,0,0)$ and $(-1,1,-1,0,0,0,1,0,0,0)$, are circuits and in the kernel of A and the first two rows of A, $(1,0,0,0,0,-1,1,0,0,1)$ and $(0,1,0,0,0,1,-1,1,0,0)$, are cocircuits and in the kernel of A^*.

d). Let $M = U_2^4$ denote the four point line. Then all 3 element subsets are circuits as well as cocircuits. Thus M is not orientable in Minty's sense.

Exercise 1. *F_7 is not orientable in Minty's sense.*

The matroids orientable in the sense of Minty form a particularly nice class of matroids which is called the class of *regular matroids*.

Theorem 1 ([11, 6])**.** *The following statements are pairwise equivalent:*

a). M is orientable in Minty's sense.

b). M is representable over any field.

c). M is representable over $GF(2)$ and also representable over some field of characteristic different from 2.

d). M is representable over the reals by a totally unimodular matrix *A (all square submatrices have determinant $0, 1$ or -1.)*

e). M has no minor isomorphic to U_2^4, F_7 and F_7^*.

We call $x, y \in \mathbb{R}^n$ *compatible* if $\forall i = 1, \ldots, n : x_i y_i \geq 0$. If x, y are compatible, then a vector $z = x + y$ is the *conformal sum* of x and y. If $L \subseteq \mathbb{R}^n$ is a vector space we call an $x \in L \setminus \{0\}$ *elementary*, if L does not contain a non-zero vector whose support is a proper subset of the support of x.

Theorem 2. *If A denotes a $\{0, 1, -1\}$ matrix representing a regular (oriented) matroid M over the reals, then* *a). the elementary vectors of* $\ker(A)$ *are scalar multiples of characteristic vectors of signed circuits of* M*.*

b). the elementary vectors of $\mathrm{im}(A^\top)$ *are scalar multiples of characteristic vectors of signed cocircuits of* M*.*

c). any integer vector $x \in \ker(A)$ *is the conformal sum of characteristic vectors of signed circuits.*

Proof. a). Let x be elementary and $x_i \neq 0$. We may assume wlog. that $i = 1$ and $x_i = 1$. Then x is a solution of the system $Ax = 0$ and $x_1 = 1$. Now the claim follows from Cramer's rule and the fact that A is totally unimodular.

b). follows using duality

c). We proceed by induction over the support of x. If x is elementary, the claim follows from part a). Otherwise, we have the characteristic vector y of a signed circuit the support of which is strictly contained in the support of x. If y is compatible with x, let $\alpha = \min\{|x_i| \mid y_i \neq 0\}$. Then by applying the inductive hypothesis to $x - \alpha y$ the claim follows. Otherwise, let $\alpha = \min\{|x_i| \mid x_i y_i < 0\}$. Then $z = x + \alpha y$ is compatible with x and of smaller support. By induction we find an elementary vector $\tilde{y}$ compatible with z and hence also with x and the claim follows as in the first case.

□

We end this section with a famous result of Paul Seymour, which states that regular matroid are not much more than graphic and cographic matroids, i.e. polygon and bond matroids.

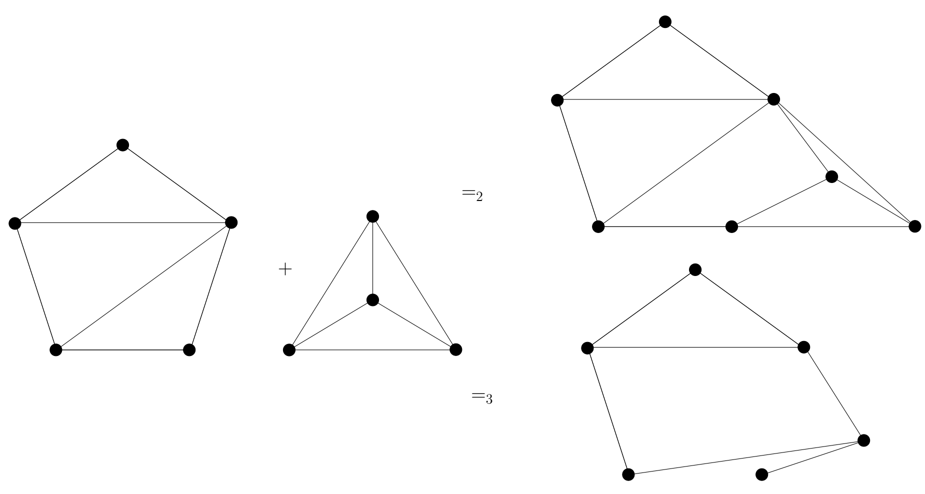


Figure 6: 2-sum and 3-sum

Theorem 3 ([10]). *Every regular matroid can be iteratively constructed using direct sums, 2-sums and 3-sums, starting with graphic matroids, cographic matroids and R_{10}.*

3 Oriented Matroids and some of their Cryptomorphisms

We have learned that an oriented matroid in Minty's sense arises from the kernel and the row space $\text{im}(A^\top)$ of a totally unimodular matrix A, where the circuits can be derived from the elementary vectors of the kernel of A and the cocircuits from the row space of A. Note, that $\ker(A)$ and $\text{im}(A^\top)$ are orthocomplementary vector spaces. The four point line is not orientable in the sense of Minty, but there seems to be an easy way to fix this.

Example 2. *The following matrix A represents the four point line over the reals:*

$$A = \begin{pmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & 2 \end{pmatrix}.$$

If we consider only the signs of elementary vectors in the kernel of this matrix we get $(+,+,-,0), (+,+,0,-), (+,0,-,+), (0,+,+,-)$ and their negatives. In the row space we find $(0,+,+,+), (+,0,+,+), (-,+,0,+), (+,-,+,0)$ and their negatives.

We observe that, if the support of a circuit and a cocircuit is nonempty, then we must have at least one coordinate where the signs coincide as well as a coordinate where they differ. This holds, doing such a construction, for arbitrary real matrices A. A vector x in its kernel and a vector y in the row space are orthogonal, hence $x^\top y = 0$ and the non-zero terms in the sum of the scalar product must not have all the same sign.

Definition 1. *Let M be a matroid on a finite set E, let $\mathcal{C}$ denote its set of circuits and $\mathcal{D}$ denote its set of cocircuits. We call M* orientable *if there is an (ordered) partition of all circuits $C \in \mathcal{C}$ into (C^+, C^-) and of all cocircuits $D \in \mathcal{D}$ into (D^+, D^-) such that $\forall C \in \mathcal{C}\, \forall D \in \mathcal{D}$:*

$$(C^+ \cap D^+) \cup (C^- \cap D^-) \neq \emptyset \iff (C^+ \cap D^-) \cup (C^- \cap D^+) \neq \emptyset. \qquad (2)$$

Again, the order of the partition does not really matter.

Proposition 1. *All matroids representable over the reals are orientable.*

Proof. Let A denote a representation matrix. Then $\ker(A) = \text{im}(A^\top)^\perp$. □

Exercise 2. *F_7 is not orientable.*

Many of the cryptomorphic axiom systems for matroids have oriented counterparts. In the following we will present some of these pairs. Let E be a finite set. By 2^E we denote the set of all subsets of E.

3.1 Circuits

We start recalling the circuit axioms of matroid theory.

Theorem 4. *A family $\mathcal{C} \subseteq 2^E$ is the set of circuits of a matroid if and only if*

C1: $\emptyset \notin \mathcal{C}$.

C2: $C_1, C_2 \in \mathcal{C}$ *and* $C_1 \subseteq C_2 \Rightarrow C_1 = C_2$.

C3: $C_1 \neq C_2 \in \mathcal{C}$ *and* $e \in C_1 \cap C_2 \Rightarrow \exists C_3 \in \mathcal{C} : C_3 \subseteq (C_1 \cup C_2) \setminus \{e\}$.

For the oriented counterpart, which is easy to visualize using directed graphs, we need some more notation. A *signed subset* C of E is a subset of E together with an ordered partition $C = (C^+, C^-)$, we denote the underlying set by $\underline{C}$. The set of all signed subsets of E is denoted by $2^{\pm E}$ and $-C = (C^-, C^+)$. The *separator* $\mathrm{sep}(C_1, C_2)$ of two signed subsets of E is defined as

$$\mathrm{sep}(C_1, C_2) = (C_1^+ \cap C_2^-) \cup (C_1^- \cap C_2^+).$$

Theorem 5. *A family $\mathcal{C} \subseteq 2^{\pm E}$ is the set of signed circuits of an oriented matroid if and only if*

C1: $\emptyset \notin \mathcal{C}$; $C \in \mathcal{C} \Rightarrow -C \in \mathcal{C}$.

C2: $C_1, C_2 \in \mathcal{C}$ *and* $\underline{C_1} \subseteq \underline{C_2} \Rightarrow C_1 \in \{\pm C_2\}$.

C3: $C_1 \neq \pm C_2 \in \mathcal{C}$ *and* $e \in \mathrm{sep}(C_1, C_2) \Rightarrow \exists C_3 \in \mathcal{C} : C_3^+ \subseteq (C_1^+ \cup C_2^+) \setminus \{e\}$ *and* $C_3^- \subseteq (C_1^- \cup C_2^-) \setminus \{e\}$.

Example 3. *a). All oriented matroids in the sense of Minty are orientable.*

b). Let $S \subseteq \mathbb{R}^n$ be a finite set of points. Consider the matroid defined by affine dependency, if i.e. $s_1, \ldots, s_n$ are dependent if there exist $\lambda_1, \ldots, \lambda_n \in \mathbb{R}$, not all equal to zero, such that $\sum_{i=1}^n \lambda_i s_i = 0$ and $\sum_{i=1}^n \lambda_i = 0$. If $C \subseteq S$ is a minimally dependent set, then there is a unique partition $C = (C^+, C^-)$ defined by the signs of the coefficients, called the Radon partition *of the points (see Figure 7 and Exercise 3). It is the unique partition of a minimal affinely independent set such that the intersection of the convex hull of both classes is non-empty. This is easily seen to fulfill the circuit axioms of oriented matroid theory.*

Figure 7: The Radon partitions of two point sets in the plane

c). Let $\vec{H}_1, \ldots, \vec{H}_n \subseteq \mathbb{R}^n$ denote a finite set of oriented hyperplanes, *i.e. codimension 1 subspaces H_i of $\mathbb{R}^n$ together with a choice of a* positive halfspace, *namely one of the two components of $\mathbb{R}^n \setminus H_i$. Assume that $\bigcap_{i=1}^n H_i = \{0\}$. We may consider the oriented hyperplanes as linear forms (up to positive scaling) in the dual space. Each one-dimensional subspace of this hyperplane arrangement, defined by a (minimal) set $D \subseteq \{1, \ldots, n\}$ such that $\dim(\bigcap_{i \notin D}^n H_i) = 1$ gives rise to two* rays. *On the other hand, $\{i \mid i \notin D\}$ is a hyperplane of the matroid defined in the dual space. Hence D is a cocircuit and each of two rays defines a partition of D depending whether it intersects the positive or the negative halfspace of H_i for $i \in D$. If we intersect this arrangement with an affine hyperplane not containing zero we get a picture like in Figure 8. The shaded triangle is on the positive side of all hyperplanes and we have indicated the cocircuits $V = (++0+-0) = (\{1,2,4\},\{5\})$ and $W = (0++0++) = (\{2,3,5,6\},\emptyset)$. Clearly, these cocircuits satisfy the circuit axioms of oriented matroid theory.*

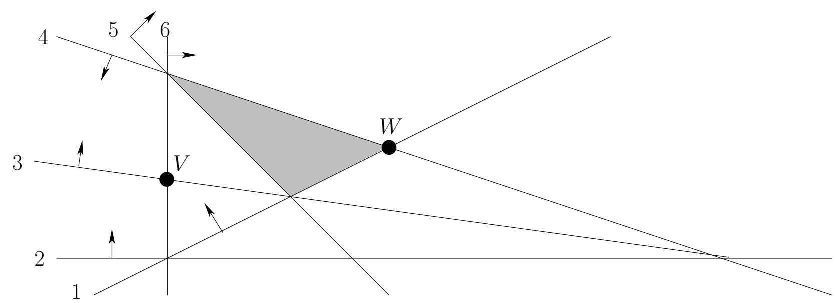


Figure 8: An affine hyperplane arrangement

Exercise 3. *If $C \subseteq S$ is a minimally affinely dependent set, then there is a unique partition $C = (C^+, C^-)$ defined by the signs of the coefficients, called the* Radon partition *such that the intersection of the convex hulls of C^+ and C^- is non-empty.*

3.2 Bases

For our purposes we prefer the basis axioms of matroid theory in its symmetric form (with double swaps):

Theorem 6. *Let $\mathcal{B} \subset 2^E$. Then $\mathcal{B}$ is the collection of bases of a matroid if and only if*

B1: $\mathcal{B} \neq \emptyset$

B2: $\forall B_1, B_2 \in \mathcal{B} \forall e \in B_1 \setminus B_2 \, \exists f \in B_2 \setminus B_1 :$

$$B_1 - e + f \in \mathcal{B} \text{ and } B_2 + e - f \in \mathcal{B}.$$

In Euclidean geometry one usually considers oriented bases according to the sign of their determinant. From a matroid theory point of view the only thing that matters is whether the determinant is zero or not. Hence, using the fact that a d element subset $B \subset (\mathbb{R}^n)^d$ is a basis iff $\det(B) \neq 0$, and if we denote the set of all d-element subsets of E by $\binom{E}{d}$, we may rephrase Theorem 6 as

Theorem 7. *Let $\chi : \binom{E}{d} \to \{0, 1\}$ be a map that is not identically zero. Then χ is the* determinant function *of a matroid of rank d if and only if $\forall B_1, B_2 \in \binom{E}{d}$ with $\chi(B_1)\chi(B_2) = 1$ and $\forall e \in B_1$*

$$\exists f \in B_2 : \chi(B_1 - e + f)\chi(B_2 + e - f) = 1.$$

The determinant function of an oriented matroid should be an alternating map to $\{0, 1, -1\}$, and we should consider ordered bases. In order to derive an oriented version of Theorem 7 we consider the Grassmann-Plücker identity:

Theorem 8. *Let $a_1, \dots, a_n, b_1, \dots, b_n \in \mathbb{R}^n$, where $b_1, \dots, b_n$ are a basis of $\mathbb{R}^n$. Define $B^i = (b_1, \dots, b_{i-1}, a_1, b_{i+1}, \dots, b_n) \in \mathbb{R}^{n \times n}$ and $A_i = (b_i, a_2, \dots, a_n)$. Then*

$$|A| \cdot |B| = \sum_{i=1}^{n} |A_i| \cdot |B^i|. \tag{3}$$

Proof. By Cramer's rule $a_1 = \sum_{i=1}^n \frac{|B^i|}{|B|} b_i$ and hence

$$\begin{aligned} |A| \cdot |B| &= \left| \sum_{i=1}^{n} \frac{|B^i|}{|B|} b_i, a_2, \dots, a_n \right| \cdot |B| \\ &= \sum_{i=1}^{n} |B^i| \, |b_i, a_2, \dots, a_n| = \sum_{i=1}^{n} |A_i| \cdot |B^i|. \end{aligned}$$

□

This pattern generalizes to arbitrary oriented matroids. If $\chi(B_1)\chi(B_2) \neq 0$ some product arising from the double swaps has to have the same sign.

Theorem 9. *Let $\chi : E^d \to \{0, 1, -1\}$ be a map that is not identically zero. Then χ is the* determinant function *of an oriented matroid of rank d if and only if*

B1: *χ is alternating, i.e. for every permutation σ*

$$\chi(x_1, \dots, x_d) = \operatorname{sign}(\sigma)\chi(x_{\sigma(1)}, \dots, x_{\sigma(d)}).$$

B2: $\forall B_1 = (x_1, \dots, x_d)$, $B_2 = (y_1, \dots, y_d) \in E^d$ *with* $\chi(B_1)\chi(B_2) \neq 0$
$\exists i \in \{1, \dots, d\}$

$$\chi(B_1)\chi(B_2) = \chi(y_i, x_2, x_3, \dots, x_d)\chi(y_1, \dots, y_{i-1}, x_1, y_{i+1}, \dots, y_d).$$

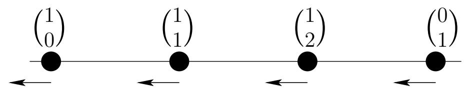


Figure 9: The configuration from Example 2

The determinant function of an oriented matroid is unique up to the sign, i.e. χ and $-\chi$ encode the same oriented matroid.

Example 4. *Recall Example 2 see Figure 9. We choose the two disjoint bases* $B_1 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ *and* $B_2 = \begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix}$. *Then* $\det(B_1) = \det(B_2) = 1$. *Let* $e = \binom{1}{0}$ *and compute the possible double swaps:*

$$\begin{vmatrix} 1 & 0 \\ 1 & 1 \end{vmatrix} \cdot \begin{vmatrix} 1 & 1 \\ 0 & 2 \end{vmatrix} = 2 \qquad \begin{vmatrix} 1 & 0 \\ 2 & 1 \end{vmatrix} \cdot \begin{vmatrix} 1 & 1 \\ 1 & 0 \end{vmatrix} = -1.$$

The Grassman-Plücker identity is $1 = 2 - 1$. We can also interpret the Grassmann-Plücker identity as orthogonality relation for circuits and cocircuits. The signs of the circuits in our example can be read of as follows. The point in the middle of three points has a different sign than the other two. The cocircuits are the complements of hyperplanes (i.e. the points), and the sign is positive if the vector points towards the point, negative otherwise.

One of the two signed circuits formed by $\begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 2 \end{pmatrix}$ has the sign pattern $(-, 0, +, -)$. It follows from Cramer's rule and Laplacian expansion that we may interpret this as

$$\left((-1)^1 \operatorname{sign}\left(\begin{vmatrix} 1 & 1 \\ 1 & 2 \end{vmatrix}\right), 0, (-1)^2 \operatorname{sign}\left(\begin{vmatrix} 1 & 1 \\ 0 & 2 \end{vmatrix}\right), (-1)^3 \operatorname{sign}\left(\begin{vmatrix} 1 & 1 \\ 0 & 1 \end{vmatrix}\right)\right).$$

On the other hand we have the signed cocircuit $(+, 0, +, +)$ which we compute as

$$\left(\operatorname{sign}\left(\begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix}\right), \operatorname{sign}\left(\begin{vmatrix} 0 & 0 \\ 1 & 1 \end{vmatrix}\right), \operatorname{sign}\left(\begin{vmatrix} 1 & 0 \\ 1 & 1 \end{vmatrix}\right), \operatorname{sign}\left(\begin{vmatrix} 1 & 0 \\ 2 & 1 \end{vmatrix}\right)\right).$$

Using the alternating sign of the determinant the orthogonality of this pair of circuit and cocircuit yields the Grassmann-Plücker identity. This generalizes to oriented matroids, since we can compute the signed circuits and cocircuits from the determinant function as follows:

Theorem 10. *Let χ be a determinant function of an oriented matroid of rank d, $C \in \mathcal{C}$ and $D \in \mathcal{D}$. Let $C \subseteq \{x_1, \ldots, x_{d+1}\}$ and $r(\{x_1, \ldots, x_{d+1}\}) = d$ where r denotes the rank function of the underlying matroid and let $y_1, \ldots, y_{d-1}$ be a basis of the hyperplane $E \setminus \underline{D}$. Then*

a).
$$C_e = \begin{cases} \epsilon(-1)^i \chi(x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_{d+1}\} & \text{if } e = x_i \\ 0 & \text{otherwise,} \end{cases}$$

where ϵ is either 1 *or* -1.

b). $D_e = \epsilon\chi(e, y_1, \ldots, y_{d-1})$, *where ϵ is either* 1 *or* -1.

3.3 Closure

The closure operator for oriented matroids is not that common. In order to define it, we will even have to change our notation slightly. Before doing so, we recall the closure operator of a matroid.

Theorem 11. *A function* $\mathrm{cl} : 2^E \to 2^E$ *is the closure operator of a matroid if and only if*

CL1: $\forall X \in 2^E : X \subseteq \mathrm{cl}(X)$

CL2: $X \subseteq Y \Rightarrow \mathrm{cl}(X) \subseteq \mathrm{cl}(Y)$

CL3: $\forall X \in 2^E : \mathrm{cl}(\mathrm{cl}(X)) = \mathrm{cl}(X)$

CL4: $y \in \mathrm{cl}(X \cup x) \setminus \mathrm{cl}(X) \Rightarrow x \in \mathrm{cl}(X \cup y)$.

Furthermore,

Proposition 2. $\mathrm{cl}(X) = X \cup \{e \in E \mid \exists C \in \mathcal{C} : e \in C \subseteq (X \cup e)\}$,

i.e. the closure adds to X all elements which close a circuit with elements from X. For oriented matroids we need an oriented version of that fact. For that purpose we assume now that we have two copies of each element, a positive and a negative one. We will denote our groundset by $\pm E$. Our signed sets C are now subsets of $2^{\pm E}$ with the property that $C \cap -C = \emptyset$.

The convex closure then is defined as

$$\mathrm{cl}(X) = X \cup \{\sigma e \in \pm E \mid \exists C \in \mathcal{C} : -\sigma e \in C \subseteq (X \cup -\sigma e)\},$$

where $\sigma \in \{+, -\}$.

Theorem 12 ([4])**.** *A function* $\mathrm{cl} : 2^{\pm E} \to 2^{\pm E}$ *is the* convex closure operator of an oriented matroid *if and only if*

CL1: $\mathrm{cl}(\emptyset) = \emptyset$.

CL2: $A \subseteq \mathrm{cl}(A) = \mathrm{cl}(\mathrm{cl}(A))$.

CL3: $A \subseteq B \Rightarrow \mathrm{cl}(A) \subseteq \mathrm{cl}(B)$.

CL4: $\mathrm{cl}(-A) = -\mathrm{cl}(A)$.

CL5: $\sigma e \in \mathrm{cl}(A \cup -\sigma e) \Rightarrow \sigma e \in \mathrm{cl}(A)$.

CL6: $\sigma e \in \mathrm{cl}(A \cup -\tau f)$ *and* $\sigma e \notin \mathrm{cl}(A) \Rightarrow \tau f \in \mathrm{cl}(A \setminus \tau f \cup -\sigma e)$.

Here $\sigma, \tau \in \{+, -\}$.

The last axiom is depicted in Figure 10. The hull operator used there is the conic hull which is more appropriate here. We used the term convex closure only for historic reasons.

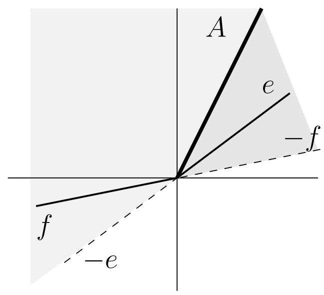


Figure 10: Axiom **CL6**

3.4 Open Sets and Covectors

The most frequently used axioms for oriented matroids are the covector axioms. Their matroid theoretic counterpart is less known. We call a set $O \subseteq E$ open, if its complement is closed, i.e. if $\mathrm{cl}(E \setminus O) = E \setminus O$. One may also consider open sets as unions of cocircuits. Then

Theorem 13. *A family $\mathcal{O} \in 2^E$ is the family of open sets of a matroid if and only if*

O1: $\emptyset \in \mathcal{O}$.

O2: $\forall O_1, O_2 \in \mathcal{O} : O_1 \cup O_2 \in \mathcal{O}$.

O3: $\forall O_1, O_2 \in \mathcal{O}\, \forall x \in O_1 \cap O_2\, \exists O_3 \in \mathcal{O}$:

$$(O_1 \cup O_2) \setminus (O_1 \cap O_2) \subseteq O_3 \subseteq (O_1 \cup O_2) \setminus \{x\}.$$

For an oriented version of the open set axioms, Figure 8 gives a guideline. The smallest open sets are the cocircuits. The union of two cocircuits $V \circ W$ should be the cell of the arrangement that we get, when we perturb V slightly in the direction of W. Hence in that example $V \circ W = (+ + + + -+)$ and $W \circ V = (+ + + + ++)$. For that purpose we define the *composition* of two signed vectors U, V as

$$(U \circ V)_e = \begin{cases} U_e & \text{if } U_e \neq 0 \\ V_e & \text{otherwise.} \end{cases}$$

We call a signed vector that is the composition of a (possibly empty) sequence of cocircuits a *covector*. The empty composition yields the all zero vector, resp. the empty signed set. Now we can give the covector axioms.

Theorem 14. *A set $\mathcal{O}$ is the set of covectors of an oriented matroid if and only if*

V0: $\emptyset \in \mathcal{O}$

V1: $\mathcal{O} = -\mathcal{O}$.

V2: $\forall U, V \in \mathcal{O} : U \circ V \in \mathcal{O}$.

V3: $\forall U, V \in \mathcal{O}\, \forall e \in \mathrm{sep}(U, V)\, \exists W \in \mathcal{O} : W^+ \subseteq (U^+ \cup V^+) \setminus \{e\}$,
$W^- \subseteq (U^- \cup V^-) \setminus \{e\}$, *and* $\forall f \notin \mathrm{sep}(U, V) : W_f = (U \circ V)_f$.

In the central hyperplane arrangement from Example 3 c) the covectors correspond exactly to the cells in the conic decomposition defined by the hyperplanes. We gave already an interpretation of V2. To visualize V3 consider a straight line connecting the two cells U and V. Since e separates the two cells, this line has to pass the hyperplane H_e. The signed vector of this intersection yields W as desired.

We close this section with an analogue of Theorem 2 c). If $\underline{X} \subseteq \underline{Y}$ and $\mathrm{sep}(X, Y) = \emptyset$, we say X *conforms to* Y and write $X \preceq Y$. We call a composition $X_1 \circ \ldots \circ X_k$ a *conformal sum* if $\forall 1 \le i, j \le k : \mathrm{sep}(X_i, X_j) = \emptyset$.

Proposition 3. *Let $\mathcal{O}$ be the set of covectors of an oriented matroid and $X \in \mathcal{O}$. Then X is the conformal sum of cocircuits of $\mathcal{O}$.*

Proof. We proceed by induction on the size of the support of X. If $\underline{X} = \emptyset$ or X is elementary, there is nothing to prove. Otherwise let D be a cocircuit and $\underline{D} \subset \underline{X}$. If $D \preceq X$ then by applying V3 to X and $-D$ (possibly repeatedly) we find a covector $Z \preceq X$ such that $\underline{Z} \subset \underline{X}$ and $\underline{Z} \cup \underline{D} = \underline{X}$ and the claim follows by induction. The case $-D \preceq X$ is analogous. Otherwise, $\mathrm{sep}(D, X) \neq \emptyset \neq \mathrm{sep}(-D, X)$ and eleminating on the separator between both pairs (possibly repeatedly) we find $Z_1, Z_2 \preceq X$ such that $\underline{Z_i} \subset \underline{X}$ and $\underline{Z_1} \cup \underline{Z_2} = \underline{X}$ and the claim follows by induction. □

4 The Topological Representation Theorem and Pseudoline Arrangements

Example 3 c) is in a certain sense the general case. Different from ordinary matroids, where sometimes geometric models seem to be a little artificial, oriented matroids always do represent a geometric situation. Since not all oriented matroids are realizable, i.e. they are given as in Example 3 c), we have to allow some small perturbations within the hyperplanes. To make this more precise, first we remove some redundancy from our hyperplane arrangement. Namely if we intersect it with the standard sphere S^{n-1} we get a cell decomposition of the sphere, where the cells correspond to the covectors of the matroid. The hyperplanes now have become hyperspheres, i.e. codimension one linear subspheres of S^{n-1}. For our general model we replace these linear hyperspheres by topological hyperspheres, but require that their intersection locally behaves like the intersection of linear spheres.

Definition 2. *Let* $S^d = \{x \in \mathbb{R}^{d+1} \mid \|x\| = 1\}$ *denote the d-dimensional* standard sphere. *A* pseudosphere of S^d *is any homeomorphic image* S *of* S^{d-1}

in S^d i.e. the image of a homeomorphism

$$\varphi : S^{d-1} \to S^d.$$

The two connected components of $S^d \setminus S$ are the sides *of S. An* oriented pseudosphere *is a pseudo hypersphere together with a choice of one of its sides.*

A finite family $\mathcal{A} = (S_e)_{e\in E}$ of pseudospheres of S^d indexed by E is an arrangement of pseudospheres *if*

A1: *$\forall A \subseteq E : S_A := \bigcap_{e\in A} S_e$ is a topological sphere.*

A2: *If $S_A \not\subseteq S_e$ and S_e^+, S_e^- are the sides of S_e, then $S_A \cap S_e$ is a codimension one topological subsphere of S_A with sides $S_A \cap S_e^+, S_A \cap S_e^-$.*

If the pseudospheres are oriented, we call the arrangement a signed arrangement of pseudospheres.

Recall the following theorem about geometric lattices, (i.e. about simple matroids):

Theorem 15. *A finite atomic lattice L is geometric if and only if for all atoms $p \in L$ and all $\ell \in L$ either $p \leq \ell$ or $\ell \vee p$ covers ℓ.*

From this it is immediate that A2 defines a matroid on E. If $\mathcal{A}$ is a signed arrangement, then the cell decomposition of the sphere induced by $\mathcal{A}$ defines a set $\mathcal{O}(\mathcal{A})$ of signed vectors.

In fact this is just another axiom system for oriented matroids:

Theorem 16 (Topological Representation, Folkman and Lawrence 1978)**.** *Let $\mathcal{O}$ denote a set of signed vectors. Then the following conditions are equivalent:*

a). $\mathcal{O}$ is a set of covectors of a loopless oriented matroid of rank $d+1$.

b). $\mathcal{O} = \mathcal{O}(\mathcal{A})$ for some signed pseudosphere arrangement $\mathcal{A}$ on S^d.

If $I \subseteq E$ and if we change the signs in all covectors $\mathcal{O}$ in entries in I, interchanging $+$ and $-$ but leaving 0, we get another oriented matroid denoted by ${}_I\mathcal{O}$ and say that ${}_I\mathcal{O}$ arises from $\mathcal{O}$ by reorientation of I. Clearly, $\mathcal{O}$ and ${}_I\mathcal{O}$ are represented by the same pseudosphere arrangement just with a different orientation. Being connected by reorientation is an equivalence relation and hence a pseudosphere arrangement corresponds to a *reorientation class* of oriented matroids.

4.1 Pseudoline Arrangements

By the topological representation theorem a rank 3 oriented matroid has a representation as a pseudocircle arrangement on S^2. If we cut this in half we get an oriented arrangement of pseudolines. The theory of pseudoline arrangements, introduced by Levi in 1926, coincides (up to considering orientation) with the theory of oriented matroids of rank 3.

Since the polygon matroid of the K_4 is of rank 3, there must be some line arrangement representing it, namely the arrangement in Figure 8. It corresponds to the orientation of the K_4 as given in Figure 11 to the left. To the right we have cut the sphere at a different place. Note, that the corresponding arrangements are *projectively equivalent*, vertex V in Figure 8 corresponds to the vertex with the open circle in the arrangement in Figure 11.

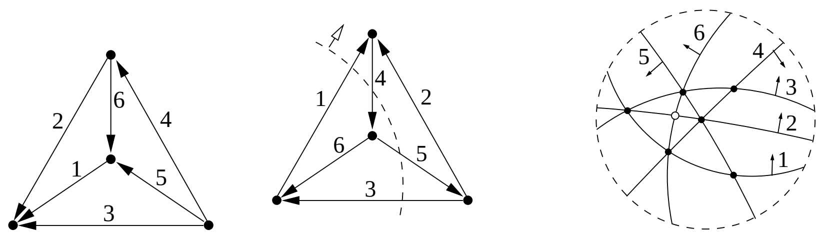


Figure 11: The orientation of K_4 corresponding to Figure 8 and another acyclic orientation with its pseudoline configuration, the cut corresponds to the vertex with the open circle.

In general oriented matroids with the same underlying matroid need not form a single reorientation class. In particular uniform matroids have many different reorientation classes. Figure 12 depicts the four reorientation classes of U_3^6.

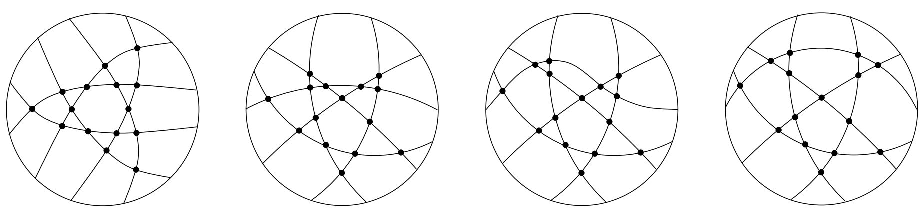


Figure 12: The four reorientation classes of U_3^6.

Already pseudoline arrangements provide many non-realizable matroids. Non-realizabilty here means, that the arrangement cannot be defined by straight lines without changing the oriented matroid. In Figure 13 we sketched two pseudoline arrangements violating Pappus' Theorem respectively Desargues' Theorem.

We end this section with some remarks on matroids with unique reorientation classes.

While already uniform oriented matroids of rank three have exponentially many reorientation classes, regular matroids have just one.

Theorem 17 (Bland, Las Vergnas 1978)**.** *Regular matroids have a unique reorientation class.*

In this field the following conjecture is still open.

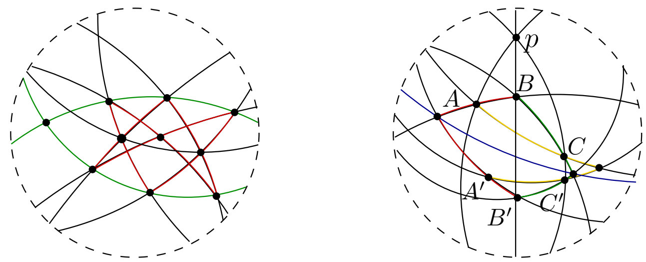


Figure 13: A non-Pappus and a non-Desargues configuration

Conjecture 1 (Las Vergnas et al. 1991)**.** *The affine d-cube has a unique reorientation class.*

The conjecture is known to hold true for $d \leq 4$.

5 Polyhedral Theory and Oriented Matroid Programming

Recall that we defined the relation $X \preceq Y$ if $\underline{X} \subseteq \underline{Y}$ and $\mathrm{sep}(X,Y) = \emptyset$. Obviously, this defines a partial order. If we add an artificial 1 to the poset it becomes a lattice, called the *face lattice* of the oriented matroid. This is, why oriented matroids are a tool to work in polyhedral theory. Every polyhedron $P = \{x \in \mathbb{R}^n \mid Ax \leq b\}$ defines an oriented matroid, where P is the vector of all plusses. For that purpose consider $C = \{\binom{x}{t} \in \mathbb{R}^{n+1} \mid (A, -b)\binom{x}{t} \leq 0\}$. This defines a hyperplane arrangement as in Example 3 c).

Several theorems from polyhedral theory have oriented matroid analogues, but some have not. Consider the cell decomposition of S^d associated with an oriented matroid of rank $d+1$. We call a d-dimensional cell of this arrangement of pseudopspheres a *simplicial tope* if it is adjacent to exactly $d+1$ cocircuits. Hence simplicial topes "are" full dimensional simplices.

Theorem 18 (Shannon 1979)**.** *Every linear arrangement of n hyperplanes, where the intersection of all hyperplanes is the empty set, contains at least n simplicial topes.*

Jürgen Richter-Gebert [8] has constructed a class of orientations of U_4^{4n} with only $3n+1$ simplicial topes. But the following is still unknown:

Conjecture 2 (Las Vergnas)**.** *Every uniform oriented matroid has at least one simplicial tope.*

Very little is known about the sphere systems associated with graphic or cographic matroids. It seems that they are very structured, i.e. from a polyhedral point of view they have a high degeneracy. The maximal cells in the sphere system of a graph correspond to the acyclic orientations, namely, if we make a cell an all positive covector, all circuits must be orthogonal to it and hence cannot be directed.

The following exercise is intended to point at some possible connections one might find between directed graph theory and oriented matroids. It may be a bit too advanced for somebody inexperienced with oriented matroids, but we sketch a solution in the last section.

Exercise 4. *Let $\mathcal{O}$ be an oriented matroid. We call a basis B of $\mathcal{O}$ a* depth first basis*, if for some orientation of $\mathcal{O}$ all fundamental circuits of B are directed.*

Let $\mathcal{O}$ be an oriented matroid and let $\mathcal{A}(\mathcal{O}^*)$ be a pseudosphere arrangement on S^d representing its dual. Then there is a bijection between the depth first bases of $\mathcal{O}$ and the antipodal pairs of simplicial topes of $\mathcal{A}(\mathcal{O}^*)$.

5.1 Oriented Matroid Programming

The most widely known offspring of oriented matroid theory is probably Bland's rule to prevent cycling in linear programming. It is a purely combinatorial rule and can be stated in oriented matroid setting. Here we will give an idea of how to translate terminology from linear programming to oriented matroids.

Given $A \in \mathbb{R}^{m\times n}$, $b \in \mathbb{R}^m$ and $c \in \mathbb{R}^n$, we call the problem

$$\max\{c^\top x \mid Ax \leq b\}$$

a *linear program*. We have already learned that the polyhedron $P = \{x \in \mathbb{R}^n \mid Ax \leq b\}$ is the all positive cell in an oriented matroid. But how to encode the linear functional and maximization? Maximization makes only sense in affine space. Oriented matroids work properly only in "two-sided projective spaces". To overcome that problem we add a pseudosphere S_g "at infinity". Our "visible world" are those covectors which are strictly positive on g (see Figure 14). The linear functional c corresponds to a family of parallel hyperplanes. These intersect in a hypersphere of S_g. Hence we may use an oriented hypersphere S_f which must intersect S_g and yields the partition

$$S_g = (S_g \cap S_f)\dot{\cup}(S_g \cap S_f^+)\dot{\cup}(S_g \cap S_f^-).$$

A direction is now a vertex (cocircuit) in S_g and a direction D is *improving* if $V \in (S_g \cap S_f^+)$. An oriented matroid program is *unbounded*, if its feasible region has a face on the positive side of infinity.

With these preparations, we can define an *oriented matroid program* as triple $(\mathcal{O}, g, f)$ meaning

$$\begin{array}{rrcll} \max & Y_f & & & \\ \text{s.t.} & Y_g & = & + & \\ & Y_e & \geq & 0 & \forall e \in E \setminus \{f,g\} \\ & Y & \in & \mathcal{O}. & \end{array}$$

Definition 3. *An* improving direction *is a covector Z such that $Z_g = 0$ and $Z_f = +$. A covector Y is* feasible *for $(\mathcal{O}, g, f)$ if $\forall e \in E \setminus \{f,g\} : Y_e \in \{0,+\}$.*

An improving direction Z is feasible *for a feasible covector Y if $Y \circ Z$ is a feasible covector. A covector Y^0 is* optimal *if there is no improving direction which is feasible for Y^0. An oriented matroid program is* unbounded *if there is a covector $Y \in \mathcal{O}$ such that $Y_g = 0$, $\forall e \in E \setminus \{f, g\} : Y_e \in \{0, +\}$ and $Y_f = +$.*

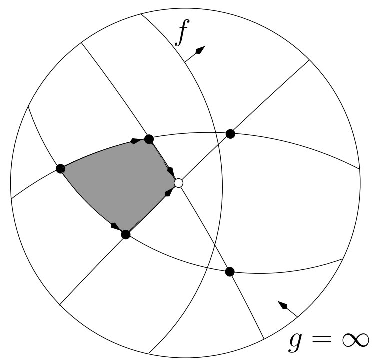


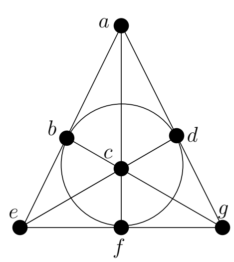
Figure 14: An oriented matroid program in rank 3

Example 5. *Consider the program depicted in Figure 14. The feasible covectors form the shaded region together with its boundary. We indicated the improving directions by putting arrows at the edges of the feasible region. Hence the white vertex is the unique optimal solution of this program.*

6 Solutions of the Exercises

Solution 1 (to Exercises 1 and 3)**.** *F_7 is not orientable.*

Proof. If an orientation of a matroid is given we may reorient it on a set I by replacing all $(C^+, C^-) \in \mathcal{C}$ with $((C^+ \setminus I) \cup (I \cap C^-), (C^- \setminus I) \cup (I \cap C^+))$ and all $(D^+, D^-) \in \mathcal{D}$ with $((D^+ \setminus I) \cup (I \cap D^-), (D^- \setminus I) \cup (I \cap D^+))$, i.e. by changing the sign in the coordinates of I in all characteristic vectors. Also changing the order of an ordered partition yields another orientation.



If F_7 were orientable, we might assume that the circuits $\{a, b, e\}, \{a, c, f\}$, and $\{a, d, g\}$ are positive, i.e. $C^- = \emptyset$. By orthogonality we have the cocircuit $(\{a\}, \{b, c, d\})$. Again by orthogonality the elements

- b, d must be in different parts of the partition of the circuit $\{b, d, f\}$,
- b, c must be in different parts of the partition of the circuit $\{b, c, g\}$,
- c, d must be in different parts of the partition of the circuit $\{c, d, e\}$.

Using this and the orientation of the three three-point lines containing a we conclude that we have cocircuits $(\{c, d\}, \{f, g\})$ and $(\{b, c\}, \{e, f\})$. Now considering the third element in $\{b, d, f\}$ we get a contradiction. $\square$

Solution 2 (to Exercise 2)**.** *If $C \subseteq S$ is a minimally affinely dependent set, then there is a unique partition $C = (C^+, C^-)$ defined by the signs of the coefficients, called the* Radon partition *such that the intersection of the convex hulls of C^+ and C^- is non-empty.*

Proof. Let $C = \{c_1, \ldots, c_k\}$, then C is affinely dependent, iff there exist $\lambda_1, \ldots, \lambda_k$, not all equal to zero, such that

$$\sum_{i=1}^{k} \lambda_i c_i = 0 \text{ and } \sum_{i=1}^{k} \lambda_i c_i = 0.$$

Set $C^+ = \{i \mid 1 \leq i \leq k \text{ and } \lambda_i > 0\}$, $C^- = \{1, \ldots, k\} \setminus I^+$. Then $\lambda := \sum_{i \in C^+} \lambda_i = \sum_{i \in C^-} -\lambda_i > 0$,

$$\sum_{i \in I^+} \frac{\lambda_i}{\lambda} c_i = \sum_{i \in I^-} \frac{-\lambda_i}{\lambda} c_i \quad \text{and} \quad \sum_{i \in I^+} \frac{\lambda_i}{\lambda} = \sum_{i \in I^-} \frac{-\lambda_i}{\lambda} = 1.$$

Hence we have found a partition such that the intersection of the convex hulls is non-empty. The same way any partition with this property gives rise to coefficients proving affine dependency. Assume, we have two partitions (C^+, C^-) and $(\tilde{C}^+, \tilde{C}^-)$. These give rise to linear combinations proving affine dependency

$$\sum_{i=1}^{k} \lambda_i c_i = 0 \text{ and } \sum_{i=1}^{k} \lambda_i c_i = 0, \qquad \sum_{i=1}^{k} \mu_i c_i = 0 \text{ and } \sum_{i=1}^{k} \mu_i c_i = 0.$$

Since these equations are invariant under linear scaling, and since all coefficients must be non-zero by minimality of the dependency, we may assume that $\lambda_1 = \mu_1 = 1$. Now,

$$\sum_{i=2}^{n} (\lambda_i - \mu_i) c_i = 0 \text{ and } \sum_{i=2}^{n} (\lambda_i - \mu_i) = 0.$$

Hence by minimality of the dependency we conclude that $\lambda_i = \mu_i$ for all $1 \leq i \leq k$. We conclude that, in particular, the partition must be unique.

$\square$

Solution 3 (to Exercise 4)**.** *There is a bijection between the depth first bases of $\mathcal{O}$ and the antipodal pairs of simplicial topes of $\mathcal{O}$.*

Proof. Let B be a depth first basis of $\mathcal{O}$ and assume that $\mathcal{O}$ is reoriented such that all fundamental circuits are positive. Let $\overline{B} = E \setminus B$ denote the corresponding cobasis. Then all fundamental cocircuits of $\overline{B}$ are directed, since the fundamental cocircuits of $\overline{B}$ coincide with the fundamental circuits of B. Hence we have $|E| - |B| = d + 1$ positive cocircuits. Since $\overline{B}$ is a basis of $\mathcal{O}$ the intersection of the corresponding pseudospheres must be empty and the orientation defines a simplex of the arrangement of $\overline{B}$ on S^d, the vertices of which correspond to the fundamental cocircuits. Since they are all positive no further pseudospere from B may intersect this simplex, hence we have found a simplicial tope.

On the other hand a simplicial tope T defines an orientation of $\mathcal{O}^*$. Let B denote the set of elements which do not define a facet of T. Then, similar to the arguments above, B is a basis of $\mathcal{O}$ and all its fundamental circuits correspond to vertices of T and hence are directed.

Note that the simplex antipodal to T defines the same bases. Instead of the determinant function χ we find $-\chi$ which give rise to the same oriented matroid. □

References

[1] Achim Bachem and Walter Kern. *Linear Programming Duality: An Introduction to Oriented Matroids.* Springer, September 1992.

[2] Anders Björner, Michel Las Vergnas, Bernd Sturmfels, Neil White, and Günter M. Ziegler. *Oriented matroids*, volume 46 of *Encyclopedia of Mathematics and its Applications.* Cambridge University Press, Cambridge, second edition, 1999.

[3] Robert G. Bland and Michel Las Vergnas. Orientability of matroids. *Journal of Combinatorial Theory Series B*, 23:94–123, 1978.

[4] J. Richard Büchi and William E. Fenton. Large convex sets in oriented matroids. *Journal of Combinatorial Theory. Series B*, 45(3):293–304, 1988.

[5] Jon Folkman and Jim Lawrence. Oriented matroids. *Journal of Combinatorial Theory. Series B*, 25(2):199–236, 1978.

[6] G.J. Minty. On the axiomatic foundations of the theories of directed linear graphs, electrical networks and network-programming. *J. Math. and Mechanics*, 15:485–520, 1966.

[7] James G. Oxley. *Matroid theory.* The Clarendon Press Oxford University Press, New York, 1992.

[8] Jürgen Richter-Gebert. Oriented matroids with few mutations. *Discrete & Computational Geometry*, 10(1):251–269, 1993.

[9] Jürgen Richter-Gebert and Günter M Ziegler. Oriented matroids. In Jacob E. Goodman and Joseph O'Rourke, editors, *Handbook of Discrete and Computational Geometry*, pages 111—132. Chapman and Hall/CRC, April 1997.

[10] P. D. Seymour. Decomposition of regular matroids. *Journal of Combinatorial Theory, Series B*, 28(3):305–359, June 1980.

[11] W.T. Tutte. A class of abelian groups. *Canad. J. Math.*, 8:13–28, 1956.