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Game-perfect Digraphs

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Abstract

In the *A-coloring game*, two players, Alice and Bob, color uncolored vertices of a given uncolored digraph D with colors from a given color set C, so that, at any time a vertex is colored, its color has to be different from the colors of its previously colored in-neighbors. Alice begins. The players move alternately, where a move of Bob consists in coloring a vertex, and a move of Alice in coloring a vertex or missing the turn . The game ends when Bob is unable to move. Alice wins if every vertex is colored at the end, otherwise Bob wins. This game is a variant of a graph coloring game proposed by Bodlaender (1991). The *A-game chromatic number* of D is the smallest cardinality of a color set C, so that Alice has a winning strategy for the game played on D with C. A digraph is *A-perfect* if, for any induced subdigraph H of D, the A-game chromatic number of H equals the size of the largest symmetric clique of H. We characterize some basic classes of A-perfect digraphs, in particular all A-perfect semiorientations of paths and cycles. This gives us, as corollaries, similar results for other games, in particular concerning the digraph version of the usual game chromatic number.
Key words: dichromatic number, digraph, game chromatic number, game-perfectness, perfect graph, quasi coloring, path, cycle

1 Introduction

Maker-breaker games on graphs are a rich subclass of the class of combinatorial games where the games define interesting graph parameters. Unlike prototypical combinatorial games like nim, these games behave even on relatively small graphs in a very complex way. In this paper, we will consider maker-breaker coloring games on directed graphs (digraphs).

Maker-breaker graph coloring games on undirected graphs have been examined since the introductory paper of Bodlaender [11]. Given a finite graph

$G = (V, E)$ and a finite color set C, two players, Alice (A) and Bob (B), alternately color a vertex of G with a color from C, so that adjacent vertices receive distinct colors. If such a move is not possible, the game ends. Alice wins if every vertex is colored at the end, otherwise Bob wins. So Alice is the maker who tries to create a proper coloring, Bob, the breaker, tries to make proper coloring impossible. Since for $|C| \geq |V|$ Alice always wins, there is a smallest number n, so that Alice has a winning strategy for the game played on a given graph G with color set C and $n = |C|$. This number n is called the *game chromatic number* $\chi_S(G)$ of G.

Even in very easy examples, e.g. if we consider the path P_4 with 4 vertices, it turns out that the game chromatic number depends on which player has the first move. Thus in order to make the game well-defined we have to fix the player moving first. We will also fix, whether either Alice or Bob or none of them has the right to miss one or several turns. This gives us six different games $S = [X, Y]$ with $X \in \{A, B\}$ moving first and $Y \in \{A, B, -\}$ having the right to miss one or several turns, where "$-$" denotes "none of the players". We denote $\mathcal{S} := \{A, B\} \times \{A, B, -\}$. Many papers on graph coloring games concentrate on the game $[A, -]$, which is denoted by g. With a slight abuse of notation, we further define the following abbreviations:

$$A := [A, A], \quad B := [B, B], \quad g_A := g := [A, -], \quad g_B := [B, -]. \tag{1}$$

Then our notation matches with most of the literature on graph coloring games. The game chromatic numbers of a graph G for the different games, as proved in [1], are related in the following way.

$$\chi_A(G) \leq \left\{ \begin{array}{lcl} \chi_g(G) & \leq & \chi_{[A,B]}(G) \\ \chi_{[B,A]}(G) & \leq & \chi_{g_B}(G) \end{array} \right\} \leq \chi_B(G). \tag{2}$$

Determining upper and lower bounds for the game chromatic numbers of several classes of graphs has received considerable attention by a large number of authors during the last two decades. Some milestones are the papers of Faigle et al. [18], Zhu [29], Kierstead [22], Cai and Zhu [14], He et al. [19], Erdös et al. [16], Sidorowicz [26], Bartnicki et al. [9], Bohman et al. [12], Wu and Zhu [27], Zhu [30], and Esperet and Zhu [17], not mentioning the many papers on relaxed or asymmetric variants of the game.

The *clique number* $\omega(G)$ of a graph G is the size of the largest complete subgraph of G. Obviously, $\omega(G) \leq \chi_S(G)$ for any game S, since the vertices of a complete graph have to be colored in distinct colors. Motivated by a question of Maria Chudnovsky, in [4] the notion of *game-perfectness* was introduced: For a game S, a graph G is *S-perfect* if, for any induced subgraph H of G, $\omega(H) = \chi_S(G)$. In [6] the classes of B-, $[A, B]$- resp. g-perfect graphs are completely characterized. The characterization of the classes of A-, $[B, A]$- resp. g_B-perfect graphs is still open, some partial results are given in [4, 6].

Game-perfectness is a special case of perfectness of a graph. Recall that a graph G is *perfect* if, for any induced subgraph H of G, the clique number of H equals the chromatic number $\chi(H)$. The *chromatic number* is the smallest number of colors in a proper coloring, i.e. the smallest n for which Alice would have a winning strategy with n colors in a one-player setting (without malicious adversary Bob). Note that $\omega(H) \leq \chi(H) \leq \chi_S(H)$ for any $S \in \mathcal{S}$, therefore S-perfect graphs are perfect in particular. Two of the most prominent among the many results in the literature concerning perfect graphs are the Weak Perfect Graph Theorem by Lovasz [23] and the Strong Perfect Graph Theorem by Chudnovsky et al. [15]; the latter gives a complete characterization of perfect graph that was conjectured by Berge [10] many years before.

Recently, the concept of perfectness has been generalized from graphs to digraphs with Hochstättler [7], using a natural generalization of coloring that defines the parameter dichromatic number introduced by Neumann-Lara [25]. In this kind of coloring the color classes do not need to be independent sets, however, they must induce acyclic digraphs, i.e. digraphs that do not contain directed cycles. This kind of coloring was reinvented by different authors and has several names, e.g. Jacob and Meyniel [20] call it *quasi coloring*. Bokal et al. [13] prove that this kind of coloring is $\mathcal{NP}$-complete even for two colors. In [7] a complete characterization of perfect digraphs is given, using the Strong Perfect Graph Theorem [15].

Theorem 1 ([7, 15])**.** *A digraph is perfect if and only if it does neither contain filled odd holes nor filled odd antiholes nor directed cycles $\vec{C}_n$, $n \geq 3$, as induced subdigraphs.*

In this paper we combine the ideas of digraph coloring and the game-theoretic approach of game-perfectness. This will lead to the concept of S-perfect digraphs. Although the notions are not yet explained, we state the main question addressed in this paper.

Problem 2. *Characterize the class of A-perfect digraphs.*

A partial answer to this question will be given. In particular we determine all A-perfect digraphs where the underlying graph is a path or a cycle.

The paper is structured as follows. In Section 2 we will define the notion of digraphs we use and explain in which way it generalizes the notion of graphs, we explain the maker-breaker digraph coloring game and introduce the concept of game-perfect digraphs. The next sections are devoted to characterize game-perfect paths and cycles. The proofs are based on some extensive case distinctions which reduce the problem to finding pure winning strategies for one of the players.

2 Preliminaries

A *digraph* D is a pair (V, A) with a finite set V of *vertices* and a set $A \subseteq \{(v, w) \in V \times V \mid v \neq w\}$ of *arcs*. An *edge* of D is a set $\{(v, w), (w, v)\} \subseteq A$, i.e. a set of two opposite arcs. A *single arc* is an arc $(v, w) \in A$, so that $(w, v) \notin A$. The *symmetric part* $S(D)$ of $D = (V, A)$ is the digraph (V, A_S), where A_S is the union of all edges; and the *oriented part* $O(D)$ of D is the digraph (V, A_O), where A_O

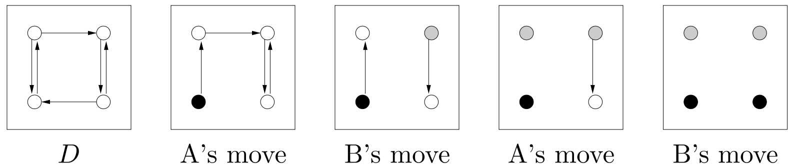


Figure 1: The g-coloring game on the digraph D to the left with two colors. After four moves, Alice wins.

is the set of all single arcs. For us, a *graph* is a symmetric digraph, i.e. a digraph D with $D = S(D)$. An *orientation of a graph* is a digraph D with $D = O(D)$. A *semiorientation* of a graph $G = (V, A_G)$ is a digraph $D = (V, A_D)$ in which for any $(v, w) \in A_G$ at least one of the arcs (v, w) and (w, v) are in A_D. A *clique* of size n or a *complete graph* K_n is a digraph $(V, \{(v, w) \in V \times V \mid v \neq w\})$ with $|V| = n$. The *clique number* $\omega(D)$ of a digraph D is the size of the largest clique of D. For other standard notation concerning digraphs (e.g. in-degree, out-degree, in-neighbor, out-neighbor, directed cycle) we refer to Bang-Jensen and Gutin [8].

By considering graphs as symmetric digraphs the usual clique number resp. chromatic number of an undirected graph coincides with the clique number resp. dichromatic number of the respective symmetric digraph, which motivates us to denote the dichromatic number of a digraph D also by $\chi(D)$. The respective parameters in the game-theoretic setting can be unified in the same way. The game we consider is the following.

We are given a digraph $D = (V, E)$ and a color set C. Two players, Alice and Bob, move alternately. A move consists in either coloring a vertex $v \in V$ with a color $c \in C$, so that no in-neighbor of v has been colored with c before (the out-neighbors are neglected), or missing the turn if this is allowed for the respective player. To make the game well-defined, we name it *S-coloring game* with $S \in \mathcal{S}$, where S has the same meaning concerning the extra rules as in the game on undirected graphs in Section 1, and we use the same abbreviations as in (1). The game ends if no further move is possible. Alice wins if every vertex is colored at the end, otherwise Bob wins.

The *S-game chromatic number* $\chi_S(D)$ of D is the smallest cardinality of a color set C, so that Alice has a winning strategy for the S-coloring game on the digraph D. The game chromatic numbers of several classes of digraphs were examined by a few authors [2, 3, 5, 28]. Note that the game chromatic number of a digraph is a completely different concept from the oriented game chromatic number introduced by Nešetřil and Sopena [24]. The digraph D is called *S-nice* if $\omega(D) = \chi_S(D)$; D is called *S-perfect* if every induced subdigraph of D is S-nice.

In order to simplify the discussion of the game we can imagine that each time a vertex v is colored every in-arc of v is deleted, since this arc does not mean any restriction for the remaining coloring moves. See Fig. 1 for a typical play of the game. Note further that by the rules of the game the color classes induce acyclic subdigraphs, therefore $\chi(D) \le \chi_S(D)$ for any digraph D. Moreover, the different versions of the game chromatic number are related in a similar way as in (2) when G is replaced by D. In particular, B-perfect digraphs are $[A, B]$-perfect, $[A, B]$-perfect digraphs are g-perfect, g-perfect digraphs are A-perfect, and A-perfect digraphs are perfect digraphs.

An *undirected path* P_n, $n \ge 2$, is a connected graph where every vertex has in-degree 2, except for exactly two vertices of in-degree 1. We also call the graph P_1, consisting of only one vertex, an *undirected path.* An *undirected cycle* C_n, $n \ge 3$, is a connected graph where every vertex has in-degree 2. In the following, by *path* resp. *cycle* we mean a semiorientation of an undirected path resp. cycle. Our main results imply that, for any $S \in \mathcal{S}$, the lists of S-perfect paths resp. cycles are finite.

3 Basic (trivial) observations

As a warm-up we note

Proposition 3. *Let $S \in \{B, g_B, [B, A]\}$. A digraph D is S-perfect if and only if it is an S-perfect graph.*

Proof. In any S-coloring game where Bob has the right of the first move, Bob wins the game with one color on the digraph $\vec{P_2}$ consisting of two vertices linked by a single arc (which has clique number 1) if he colors the vertex with out-degree 1 in his first move. Therefore an S-perfect digraph does not have single arcs, thus it is a graph. $\square$

In view of Proposition 3, in the rest of the paper we will focus on the classes of A-, g-, resp. $[A, B]$-perfect digraphs, which allow for a richer structure.

We first consider digraphs with clique number 1, i.e. orientations of graphs, which will be called *simple digraphs.* An *in-star* is a digraph with a center vertex of in-degree n and n other vertices with in-degree 0 and no further vertices. We observe

Proposition 4. *The A-perfect (resp. g-perfect resp. $[A, B]$-perfect) digraphs with clique number 1 are exactly the digraphs in which one component is an in-star and the other components are isolated vertices.*

Proof. Obviously, an in-star plus isolated vertices is S-perfect for $S \in \{A, g, [A, B]\}$: Alice colors the center of the in-star in the first move of the S-coloring game with one color, and wins. We will prove that any other simple digraph is not A-perfect, and hence, by (2), is not g- and not $[A, B]$-perfect. If a simple digraph D contains two nontrivial components, it has a subdigraph on 4 vertices with two nonadjacent arcs, which is not A-nice (cf. Lemma 5, case F_4), implying that D is not A-perfect. Now let D be a connected simple digraph which is not an in-star. We will prove that D is not A-nice (and therefore, not A-perfect).

Since D is not an in-star but connected, D has at least two vertices with in-degree of at least 1. Let v and w be such vertices. A winning strategy for Bob with 1 color is the following. If Alice, in her first move, colors a vertex z with non-zero out-degree, Bob wins, since an out-neighbor of z cannot be colored any more. Otherwise, we may assume w.l.o.g. that w has not been colored by Alice. Then Bob colors an in-neighbor of w, so that w cannot be colored any more. Thus he wins in any case. □

In the rest of the paper we will discuss some classes of digraphs with clique number 2.

4 Paths

Every subdigraph of a path is a forest of paths. Therefore we consider the hereditary class of forests of paths, i.e. of those digraphs each component of which is a path. In all figures, a straight line between two vertices v and w represents the two arcs (v, w) and (w, v). A single arc (v, w) is depicted by an arrow directed from v to w.

Lemma 5. *If a digraph D contains any of the forbidden configurations $F_{3,1}$, $F_{3,2}$, F_4, $F_{5,1}$, $F_{5,2}$, $F_{7,1}$, $F_{7,2}$, or F_8 depicted in Fig. 2 as induced subdigraph, then D is not A-perfect.*

Proof. We verify that Bob has a winning strategy in the A-coloring game with $\omega(F)$ colors on each of the forbidden configurations F of Fig. 2. We refer to the names of the vertices in the figure.

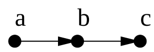
$F_{3,1}$: In order not to create a win for Bob, Alice must miss her first turn or color c. But then Bob colors a and wins with one color, since b cannot be colored feasibly any more.

$F_{3,2}$: In order not to create a win for Bob, Alice must miss her first turn or color a or c. But then Bob colors b and wins with one color, since the remaining uncolored vertex cannot be colored any more.

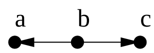
F_4: If Alice misses her turn or colors c or d, Bob colors a, and b cannot be colored any more. Otherwise, Bob colors c, so that d cannot be colored feasibly any more with one color.

$F_{5,1}$, $F_{5,2}$: Now we consider the game with two colors. If Alice colors a vertex with color 1, then Bob can color a vertex at distance 2 with color 2, which results in a win for him. If Alice misses her first turn, Bob colors c with color 1. In his second move he will color either a or e with color 2 next to an uncolored vertex. Alice can destroy only one of the two possibilities in her move. Thus Bob will win.

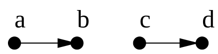
$F_{7,1}$, $F_{7,2}$: This is similar to the previous case: If Alice colors a vertex with color 1, Bob can color a vertex at distance 2 with color 2. This results in a win for him if he obeys the rule that, if Alice colors e, he has to color g.



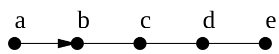
(a) The forbidden configuration $F_{3,1}$



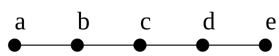
(b) The forbidden configuration $F_{3,2}$



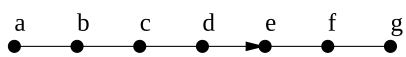
(c) The forbidden configuration F_4



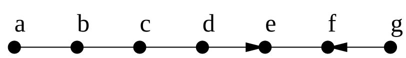
(d) The forbidden configuration $F_{5,1}$



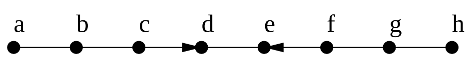
(e) The forbidden configuration $F_{5,2}$



(f) The forbidden configuration $F_{7,1}$



(g) The forbidden configuration $F_{7,2}$



(h) The forbidden configuration F_8

Figure 2: Some forbidden configurations for A-perfectness

If Alice misses her first turn, Bob colors the central vertex d with color 1. Alice, in her next move, cannot preclude Bob from coloring either vertex b or vertex f with color 2 in his second move. Again, Bob wins.

F_8: If Alice colors a vertex with color 1, Bob can color a vertex at distance 2 with color 2, so that he wins. (In case Alice colors d resp. e, he has to color f resp. c.) If Alice misses her turn, Bob colors c with color 1. In his next move he will either color a or e with color 2, in order to win. Alice can avoid only one of these situations by her next move.

Thus the forbidden configurations are not A-perfect. Then, by the definition of A-perfectness, D is not A-perfect. □

Lemma 6. *Let P be a path with $n \geq 10$ vertices. Then P is not A-perfect. Moreover, P contains a forbidden configuration as an induced subdigraph.*

Proof. Assume P is A-perfect. If P contains 3 single arcs, P has an induced $F_{3,1}$, $F_{3,2}$ or F_4. So P has at most 2 single arcs. If there are 2 single arcs then these are adjacent or at distance 1, otherwise P has an induced F_4. Since the length of the path is $n-1 \geq 9$, P contains either an induced $P_5 = F_{5,2}$, which is a forbidden configuration, or P is of the form $v_1v_2v_3v_4v_5v_6v_7v_8v_9v_{10}$, where $v_1v_2v_3v_4$ and $v_7v_8v_9v_{10}$ are (undirected) P_4's, v_5v_6 is an edge, and between v_4 and v_5 resp. between v_6 and v_7 there are single arcs. If there was an arc (v_5, v_4) or an arc (v_6, v_7), P would contain $F_{5,1}$. So there are arcs (v_4, v_5) and (v_7, v_6). But then P contains $F_{7,2}$, which is a contradiction. □

Lemma 7. *For each of the paths Q_i ($i = 0, \ldots, 47$) from Fig. 3, Alice wins the A-coloring game played on Q_i with $\omega(Q_i)$ colors.*

Proof. We describe winning strategies for Alice in the A-coloring game. Remark that in most cases Alice does not need to make use of her right to miss the first turn, furthermore these strategies are also winning strategies in the g- resp. $[A, B]$-coloring game. In some cases we note even two strategies, here Alice wins no matter which player begins, both situations will be used later. We refer to the names of the vertices in Fig. 3.

Q_0: Alice obviously wins with one color.

Q_1: Alice colors b and wins with one color.

Q_2: Here, Alice always wins with 2 colors.

Q_3, Q_4, Q_5, Q_6: Alice misses her first turn. No matter how Bob plays, after her second move she can assure that the central vertex b is colored, hence she will win.

Second (alternative) strategy: If Alice, in her first move, colors the central vertex b, she will win.

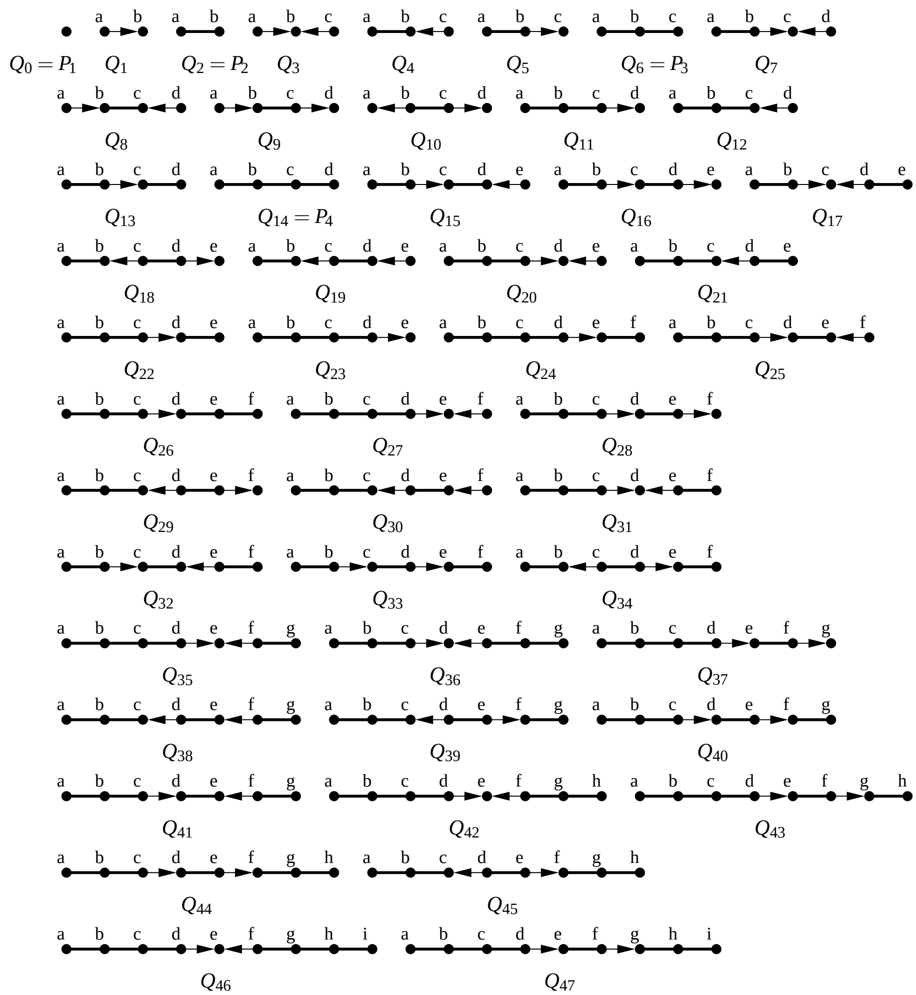


Figure 3: The 48 A-perfect paths

Q_7, Q_8, Q_9, Q_{10}, Q_{11}, Q_{12}, Q_{13}, Q_{14}: Alice's strategy is the same as in the undirected case (Q_{14}): she misses her first turn, then she colors a vertex at distance 2 to the vertex Bob has colored in the same color.

Second strategy for Q_7, Q_{13}: Alice colors c and wins.

Second strategy for Q_9, Q_{10}, Q_{11}: Alice colors d and reduces the digraph to Q_4, Q_5, or Q_6.

Q_{15}: Alice misses her first turn. If Bob then colors a with color 1, she colors d with color 2. If Bob colors b resp. d, then she colors d resp. b with the same color. If Bob colors c resp. e, she colors e resp. c with the same color. In all cases she will win now. (Note that vertex c, if colored, is no danger for vertex b.)

Q_{16}: Alice colors e and reduces the digraph to the configuration Q_{13}. On this digraph she has a winning strategy even if she misses her first turn, which means if Bob plays first, as seen above.

Q_{17}: Alice colors c and reduces the digraph to $2P_2$.

Q_{18}: Alice colors e and reduces the digraph to Q_{13}.

Q_{19}: Alice colors b and reduces the digraph essentially to Q_4.

Q_{20}: Alice colors d and reduces the digraph essentially to P_3.

Q_{21}: Alice's strategy is the same as for Q_{15} with the following permutation of vertices: $(a, e)(b, d)$. In particular she wins on this configuration even if she is the second player.

Q_{22}: Alice colors d and reduces the digraph essentially to P_3.

Q_{23}: Alice colors e and reduces the digraph to P_4.

Q_{24}: Alice colors e and reduces the digraph to a partially colored P_2 and an uncolored P_4.

Q_{25}, Q_{26}: In her first move, Alice colors b with color 1. If Bob then colors a or c, Alice replies by coloring e with color 2. If Bob colors d resp. f, Alice colors f resp. d with the same color. In case Bob colors e, Alice answers by coloring d with the other color. After that the possible colors for the uncolored vertices are fixed (except possibly for a P_1 consisting of f in case of Q_{25}).

Q_{27}: Alice colors e and reduces the digraph to $P_4 \cup P_1$.

Q_{28}: Alice colors d and reduces the digraph to a P_3 (left half) and a Q_5 with one colored vertex (right half). The latter will be colored completely in any case.

Q_{29}: Alice colors f and reduces the digraph to Q_{21}. For Q_{21}, Alice has a winning strategy when playing as second player, as seen above.

Q_{30}: Alice, in her first move, colors e. No matter what Bob does, Alice can color b or c appropriately in order to fix the coloring (except for the isolated vertex f).

Q_{31}: Alice colors d and reduces the digraph to a P_3 and a P_2.

Q_{32}: Alice colors a with color 1. If Bob then colors b or c, Alice colors d. If Bob colors d or e, Alice colors c. In case Bob colors f, Alice colors d with color 2. After that Alice will win.

Q_{33}, Q_{34}: Alice colors e and reduces the digraph essentially to Q_{13}, on which Alice wins even when playing as second player as seen above.

Q_{35}: Alice colors e and reduces the digraph to a P_4 and a P_2.

Q_{36}: Alice colors d and reduces the digraph to a $2P_3$.

Q_{37}: Alice colors e and reduces the digraph to an uncolored P_4 and a partially colored Q_5. On the latter Alice will win in any case.

Q_{38}: Alice colors e. In case Bob replies by coloring f, g, or d, Alice answers by coloring b with the second color. If Bob colors c, Alice colors b with a suitable color. If, on the other hand, Bob colors a resp. b, Alice colors c with the same resp. the other color. Then Alice will win.

Q_{39}: Alice colors f and reduces the digraph essentially to Q_{21}, on which Alice has a winning strategy as second player as seen above.

Q_{40}: Alice colors d. Now Alice has to make b and f colored as quickly as possible. She uses for her next move the strategy that, if Bob plays in the right half, she answers in the right half, otherwise in the left half. In her third move she can accomplish her goal, resulting in a win.

Q_{41}: Alice colors b with color 1. If Bob colors a, c, or g, Alice answers by coloring e with color 2. If Bob colors d resp. f, then Alice colors f resp. d with the same color. In case Bob colors e, Alice colors d with the other color. After that Alice will win.

Q_{42}: Alice colors e and reduces the digraph to a P_4 and a P_3.

Q_{43}: Alice colors e with color 1. If Bob plays in the component $\{a, b, c, d\}$, Alice answers in this component in such a way that the coloring of the component is fixed. If Bob colors f or h, Alice colors g; if Bob colors g, Alice colors f. She then misses her turn until Bob starts coloring the component $\{a, b, c, d\}$ and then follows her winning strategy for a P_4.

Q_{44}: Alice colors d. She then answers to moves of Bob in the component $\{a, b, c\}$ according to her strategy for the P_3 in the same component, and to the first move of Bob in the right component in the following way: If Bob colors e, Alice colors g with the same color. If Bob colors f resp. h, Alice colors h resp. f with the same color. If Bob colors g, Alice colors f with the other color.

Q_{45}: Alice misses her first turn. If Bob colors a or b resp. c, Alice colors c resp. b leaving a component Q_{21} at the right, on which Alice has a winning strategy as a seond player. By reasons of symmetry, a similar reasoning holds if Bob colors f, g, or h. If, on the other hand, Bob colors d with color 1, Alice colors b with color 1. If Bob, in his second move, colors a, c, or e, Alice colors g with color 2. If Bob, in his second move, colors f, Alice colors h with the same color. If Bob, in his second move, colors g or h, Alice colors f. The case that Bob, in his first move, colors e, is similar, by reasons of symmetry. In any case, after Alice's third move the coloring will be fixed.

Q_{46}: Alice colors e and reduces the digraph to a $2P_4$.

Q_{47}: Alice colors e. Now she uses a reply-strategy: if Bob plays in the left component $\{a, b, c, d\}$, she answers according to her strategy for the P_4; if Bob plays in the right component, she replies as follows: if Bob colors f or g, she colors h, if Bob colors h or i, she colors g. This will result in a win.

This completes the proof. □

Lemma 8. *The configurations of Fig. 3 are exactly those paths with at most 9 vertices which do not contain any of the forbidden configurations of Fig. 2 as induced subdigraphs.*

Proof. We enumerate all paths with at most 9 vertices and check that they contain a forbidden configuration or are one of the configurations Q_i. Note that, as in the proof of Lemma 6, we can restrict ourselves to paths with at most 2 single arcs, and if a path has two single arcs, they are adjacent or at distance 1, otherwise the path would contain a forbidden configuration $F_{3,1}$, $F_{3,2}$ or F_4. Furthermore, by the symmetry of paths, we may assume that, if there are one or two single arcs, then the barycenter of these single arcs is in the "right" half of the path. We denote the paths under consideration by an n-tuple $(x_1, \ldots, x_n)$, where $x_i = 0, 1, 2$ means that between the ith and the $(i + 1)$st vertex there is an edge, a "forward" single arc (from left to right), a backward single arc, respectively.

Using these simplifications, we distiguish four cases. First, if a path contains no single arcs, then we have $P_1 = Q_0$, $P_2 = Q_2$, $P_3 = Q_6$, $P_4 = Q_{14}$, and for $n \geq 5$ the path contains an induced $P_5 = F_{5,2}$.

Now consider the second case: the paths with exactly one single arc. See Table 1.

1	Q_1		000010	contains $F_{5,2}$
2	Q_1		000020	contains $F_{5,2}$
01	Q_5		000100	is $F_{7,1}$
02	Q_4		000200	contains $F_{5,1}$
001	Q_{11}		0000001	contains $F_{5,2}$
002	Q_{12}		0000002	contains $F_{5,2}$
010	Q_{13}		0000010	contains $F_{5,2}$
020	Q_{13}		0000020	contains $F_{5,2}$
0001	Q_{23}		0000100	contains $F_{5,2}$
0002	is $F_{5,1}$		0000200	contains $F_{5,2}$
0010	Q_{22}		0001000	contains $F_{7,1}$
0020	Q_{21}		0002000	contains $F_{5,1}$
00001	contains $F_{5,2}$		00000001	contains $F_{5,2}$
00002	contains $F_{5,2}$		00000002	contains $F_{5,2}$
00010	Q_{24}		00000010	contains $F_{5,2}$
00020	contains $F_{5,1}$		00000020	contains $F_{5,2}$
00100	Q_{26}		00000100	contains $F_{5,2}$
00200	Q_{26}		00000200	contains $F_{5,2}$
000001	contains $F_{5,2}$		00001000	contains $F_{5,2}$
000002	contains $F_{5,2}$		00002000	contains $F_{5,2}$

Table 1: Case 2: Paths with exactly one single arc

We are left with the paths with exactly two single arcs. First consider those with adjacent single arcs. In view of $F_{3,1}$ and $F_{3,2}$ we only have to consider pairs of single arcs of type $(1, 2)$. See Table 2.

12	Q_3		001200	Q_{36}
012	Q_7		0000012	contains $F_{5,2}$
0012	Q_{20}		0000120	contains $F_{5,2}$
0120	Q_{17}		0001200	Q_{42}
00012	Q_{27}		00000012	contains $F_{5,2}$
00120	Q_{31}		00000120	contains $F_{5,2}$
000012	contains $F_{5,2}$		00001200	contains $F_{5,2}$
000120	Q_{35}		00012000	Q_{46}

Table 2: Case 3: Paths with two adjacent single arcs

The last case are paths with two nonadjacent single arcs. See Table 3.

This proves the lemma. □

101	Q_9
102	Q_8
201	Q_{10}
202	Q_9
0101	Q_{16}
0102	Q_{15}
0201	Q_{18}
0202	Q_{19}
00101	Q_{28}
00102	Q_{25}
00201	Q_{29}
00202	Q_{30}
01010	Q_{33}
01020	Q_{32}
02010	Q_{34}
02020	Q_{33}
000101	Q_{37}
000102	is $F_{7,2}$
000201	contains $F_{5,1}$
000202	contains $F_{5,1}$
001010	Q_{40}
001020	Q_{41}
002010	Q_{39}
002020	Q_{38}

0000101	contains $F_{5,2}$
0000102	contains $F_{5,2}$
0000201	contains $F_{5,2}$
0000202	contains $F_{5,2}$
0001010	Q_{43}
0001020	contains $F_{7,2}$
0002010	contains $F_{5,1}$
0002020	contains $F_{5,1}$
0010100	Q_{44}
0010200	is F_8
0020100	Q_{45}
0020200	Q_{44}
00000101	contains $F_{5,2}$
00000102	contains $F_{5,2}$
00000201	contains $F_{5,2}$
00000202	contains $F_{5,2}$
00001010	contains $F_{5,2}$
00001020	contains $F_{5,2}$
00002010	contains $F_{5,2}$
00002020	contains $F_{5,2}$
00010100	Q_{47}
00010200	contains $F_{7,2}$
00020100	contains $F_{5,1}$
00020200	contains $F_{5,1}$

Table 3: Case 4: Paths with two non-adjacent single arcs

Theorem 9. *Let F be a forest of paths with components $D_1, D_2, \ldots, D_k$. Then the following statements are equivalent:*

(a) F is A-perfect.

(b) F does not contain any of the forbidden configurations $F_{3,1}$, $F_{3,2}$, F_4, $F_{5,1}$, $F_{5,2}$, $F_{7,1}$, $F_{7,2}$, or F_8 depicted in Fig. 2 as an induced subdigraph.

(c) Every component of F, except at most one, is either an undirected path P_1, P_2, P_3, or P_4, and the remaining component is one of the 48 configurations depicted in Fig. 3.

In particular, the only A-perfect paths are those depicted in Fig. 3.

Proof. By Lemma 5 we have (a) $\Longrightarrow$ (b).

Consider (b) $\Longrightarrow$ (c). Assume that F does not contain any forbidden configuration. As F_4 is forbidden, every component D_i (with at most one exception, say D_1) is a graph, i.e. an undirected path P_{n_i}. Since $P_5 = F_{5,2}$ is forbidden, $n_i \leq 4$ for all $i \geq 2$. By Lemma 6 the remaining component D_1 has at most 9 vertices. Therefore, by Lemma 8, D_1 is one of the 48 configurations of Fig. 3. Thus F is of the desired form.

Finally we prove (c) $\Longrightarrow$ (a). Assume that F is of the form as in (c). By Lemma 7, the 48 configurations of Fig. 3 are A-nice. Every digraph consisting of an arbitrary component C' which is one of the digraphs of Fig. 3 and some components which are undirected paths P_1, P_2, P_3, or P_4 is A-nice as well, as we shall see. Indeed, a winning strategy for Alice is the following: in her first move she plays on C', after that she always plays in the component on which Bob has played in his last move, in both cases according to her winning strategy for the respective components. Playing on a component possibly includes the use of Alice's right to miss a turn if this is necessary according to her winning strategy for C' or if a component is completely colored. Note that her winning strategies for P_j, $1 \leq j \leq 4$, always allow her to make Bob color the first vertex, therefore the strategy described above is feasible. Since every induced subdigraph of F is also of the type of digraphs described in (c), F is not only A-nice, but A-perfect. $\square$

Corollary 10. *Let F be a forest of paths with components $D_1, D_2, \ldots, D_k$. Then the following statements are equivalent:*

(a) F is $[A, B]$-perfect

(b) F is g-perfect.

(c) F does neither contain any of the forbidden configurations $F_{3,1}$, $F_{3,2}$, F_4, depicted in Fig. 2, nor any of the paths Q_8, Q_{12}, Q_{14}, depicted in Fig. 3, as an induced subdigraph.

(d) Every component of F, except at most one, is either an undirected path P_1, P_2, or P_3, and the remaining component is one of the 24 configurations Q_0 — Q_7, Q_9 — Q_{11}, Q_{13}, Q_{16} — Q_{20}, Q_{22}, Q_{28}, Q_{31}, Q_{33}, Q_{34}, Q_{36}, Q_{40}, depicted in Fig. 3.

Proof. By (2) we have (a) $\Longrightarrow$ (b).

Consider (b) $\Longrightarrow$ (c). Assume that F is g-perfect. By (2), F is A-perfect, therefore, by Theorem 9, F does not contain any of the forbidden configurations $F_{3,1}$, $F_{3,2}$, and F_4. Furthermore, on the paths Q_8, Q_{12}, and Q_{14}, Bob has the following winning strategy in the g-coloring game with 2 colors. After Alice has colored a vertex, Bob colors the vertex at distance 2 with a different color. This will lead to a win for him.

Consider (c) $\Longrightarrow$ (d). If F does not contain any of the six in (c) mentioned configurations, it does, in particular, not contain any of the configurations of Fig. 2 as induced subdigraphs, since $P_4 = Q_{14}$ is contained in $F_{i,j}$ with $i \in \{5,7\}$ and $j \in \{1,2\}$, and Q_8 is contained in F_8. Therefore, by Theorem 9 and since P_4 is forbidden, every component of F, except of at most one, is P_1, P_2, or P_3, and the exceptional component is one of the 48 configurations of Fig. 3. As the forbidden configuration Q_8 is contained in Q_{15}, Q_{25}, Q_{32}, and Q_{41}, the forbidden configuration Q_{12} is contained in Q_{21}, Q_{26}, Q_{29}, Q_{30}, Q_{38}, Q_{39}, Q_{44}, and Q_{45}, and the forbidden configuration Q_{14} is contained in Q_{23}, Q_{24}, Q_{27}, Q_{35}, Q_{37}, Q_{42}, Q_{43}, Q_{46}, and Q_{47}, only the 24 configurations mentioned in (d) are allowed.

Finally, consider (d) $\Longrightarrow$ (a). First note that the 24 configurations mentioned in (d) are $[A,B]$-nice. This has been proven in Lemma 7, cf. the remark at the beginning of the proof. By a similar strategy on the components as in Theorem 9, Alice wins the $[A,B]$-coloring gamne on F with 2 colors. Therefore F is $[A,B]$-nice. Since every induced subdigraph of F is also of the type of digraphs described in (d), F is not only $[A,B]$-nice, but $[A,B]$-perfect. $\square$

5 Cycles and more

Lemma 11. *Let C be a cycle with $n \geq 7$ vertices. Then C is not A-perfect.*

Proof. Assume C is A-perfect. If C has three single arcs, then it contains a forbidden configuration $F_{3,1}$, $F_{3,2}$, or F_4 as induced subdigraph. So C has at most 2 single arcs, and if there are two, then these are adjacent or at distance 1. There are remaining $m \geq n-3 \geq 4$ edges, which form a (forbidden) $P_5 = F_{5,2}$, a contradiction. $\square$

Lemma 12. *For each of the cycles O_i ($i = 1, \ldots, 8$) from Fig. 4, Bob wins the A-coloring game played on O_i with $\omega(O_i)$ colors.*

Proof. We refer to the names of the vertices in Fig. 4. A winning strategy for Bob for the O_i is as follows.

O_1: Here, we play with one color. In order not to loose directly, Alice has to miss her turn or to color c. But then Bob colors a and wins, since b cannot be colored any more.

O_2: The coloring of any vertex results in a win for Bob.

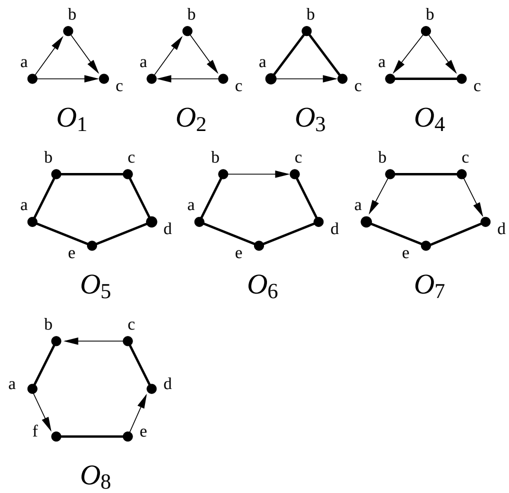


Figure 4: 8 forbidden cycles

O_3: In the following configurations we play with 2 colors. If Alice colors a, Bob colors c. If Alice colors b or c with the first color, Bob colors a with the second color. If Alice misses her turn, Bob colors a. In all cases he will win.

O_4: Coloring b results in a direct win for Bob. If Alice colors a or c or misses her turn, Bob colors b.

O_5: Since $O_5 = C_5$ is not perfect, it is in particular not A-perfect.

O_6: If Alice colors a vertex with color 1, Bob colors a vertex at distance 2 with color 2 with the following restriction: if Alice colors c, Bob colors e. If Alice misses her turn, Bob colors b. In both cases, Bob will win, in the second case by the same reason by which an undirected 5-cycle cannot be colored with 2 colors.

O_7: If Alice colors a resp. d with color 1, Bob colors d resp. a with color 2. If Alice colors b or c with color 1, Bob colors e with color 2. If Alice colors e with color 1, Bob colors b with color 2. If Alice misses her turn, Bob colors e with color 1. In his next move he will either color b with color 2 and leave a uncolored or color c with color 2 and leave d uncolored. Alice, in her intermediate move, cannot preclude Bob from doing so.

O_8: If Alice colors a with color 1, Bob colors e with color 2 and wins directly. In case Alice colors b with color 1, Bob colors e with color 1. After Alice's next move, Bob can either color c with color 2 and leave d uncolored (and thus uncolorable) or color a with color 2 and leave f uncolored (and thus uncolorable). The cases that Alice colors c, d, e, or f are in line by the

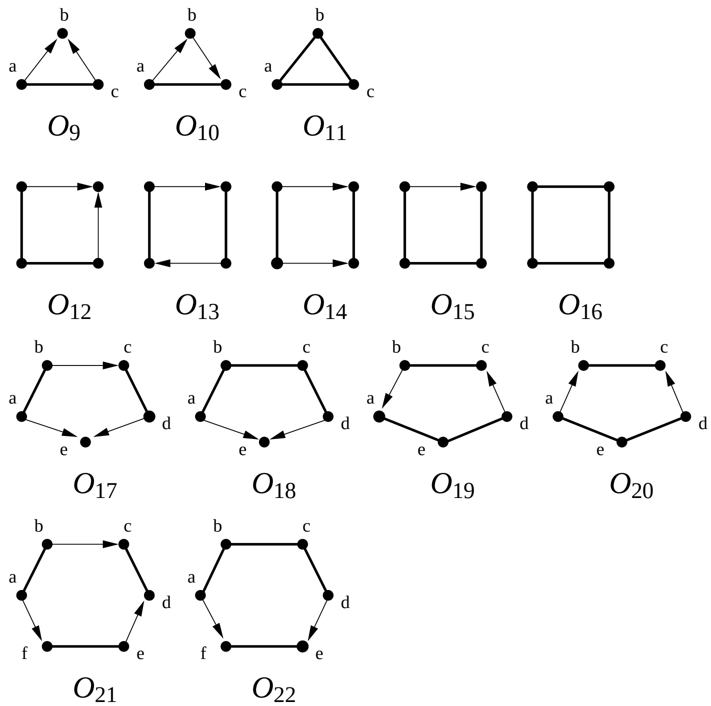


Figure 5: The 14 A-perfect semiorientations of cycles

symmetry of the digraph. If Alice misses her first turn, Bob colors a with color 1. In his next move he will color a vertex at distance 2 with color 2, leaving an uncolored vertex inbetween. Alice cannot prevent him from doing so. Note that even if Alice colors d with color 2, Bob can color e with color 2. So Bob wins in any case.

This proves the lemma. $\square$

Lemma 13. *For each of the cycles O_i ($i = 9, \ldots, 22$) from Fig. 5, Alice wins the A-coloring game played on O_i with $\omega(O_i) = 2$ colors.*

Proof. We refer to the names of the vertices in Fig. 5. A winning strategy for Alice for the O_i is as follows.

O_9: We play with two colors. Alice colors b and wins.

O_{10}: Alice colors c and wins.

O_{11}: Obviously, the K_3 can be colored with three colors.

O_{12}, O_{13}, O_{14}, O_{15}, O_{16}: In the following we play again with two colors. Alice misses her first turn. If Bob colors a vertex, Alice colors the unique vertex at distance 2 with the same color. Then, in any case, the two remaining vertices can be colored with the second color.

Second strategy for O_{12}, O_{13}: Alice colors a vertex with minimum out-degree in her first move. If then the vertex at distance 2 is not colored by Bob, Alice colors it in her second move. In any case she wins.

O_{17}, O_{18}: Alice colors e and reduces the digraph to a path Q_{13} or P_4 for which she has a winning strategy even as the second player, see Appendix B.

O_{19}: Alice colors c. If Bob colors a resp. b, Alice colors e with a different resp. the same color. If Bob colors d resp. e, Alice colors a with the same resp. the other color. In any case, Alice will win thereafter.

O_{20}: Alice colors e. By reasons of symmetry, we may assume that Bob colors a resp. b. Then Alice colors c with the same resp. the other color. After that the coloring is fixed.

O_{21}: Alice colors f. If Bob then colors a or c, Alice colors d with the other color. If Bob colors b resp. d, Alice colors d resp. b with the same color as Bob. If Bob colors colors e, Alice colors c with the same color.

O_{22}: Alice misses her first turn. Then she uses a copy cat strategy: In case Bob colors a vertex on the left side, Alice colors the corresponding vertex on the right side with the other color, and vice versa. Playing this way, she will win.

This proves the lemma. □

Lemma 14. *The configurations of Fig. 4 and Fig. 5 are exactly those cycles with at most 6 vertices which do not contain any of the forbidden configurations $F_{3,1}$, $F_{3,2}$, F_4, resp. $F_{5,1}$ of Fig. 2 as induced subdigraphs.*

Proof. Let C be an n-cycle with at most 6 vertices. The undirected cycles C_3, C_4, C_5, resp. C_6, are O_{11}, O_{16}, O_5, resp. contain a forbidden $P_5 = F_{5,2}$. So we only have to consider cycles with at least one single arc. We encode the cycle, assuming a fixed orientation and starting vertex, by an n-tuple $(x_1, \dots, x_n)$, where $x_i = 1$ denotes a forward single arc, $x_i = 2$ denotes a backward single arc, and $x_i = 0$ denotes an edge. W.l.o.g. $x_1 = 1$. We list all such tuples with the following restrictions. For $n \geq 4$, if there are three (cyclically!) consecutive single arcs, then the cycle contains a forbidden $F_{3,1}$ or $F_{3,2}$. So we leave all tuples with three consecutive single arcs away. Furthermore, if $n = 6$, we may assume that $x_4 = 0$, otherwise the cycle contains a forbidden F_4.

The restricted listing is displayed in Table 4.

This proves the lemma. □

100	O_3	10210	contains $F_{3,2}$
101	O_{10}	10220	contains $F_{3,1}$
102	O_4	11000	contains $F_{3,1}$
110	O_{10}	11010	contains $F_{3,1}$
111	O_2	11020	contains $F_{3,1}$
112	O_1	12000	O_{18}
120	O_9	12010	O_{17}
121	O_1	12020	O_{17}
122	O_1	100000	contains $F_{5,2}$
1000	O_{15}	100001	contains $F_{5,2}$
1001	contains $F_{3,1}$	100002	contains $F_{5,2}$
1002	contains $F_{3,2}$	100010	contains $F_{5,1}$
1010	O_{13}	100020	contains $F_{5,1}$
1020	O_{14}	101000	contains $F_{5,1}$
1100	contains $F_{3,1}$	101001	contains $F_{3,1}$
1200	O_{12}	101002	contains $F_{3,2}$
10000	O_6	101010	O_8
10001	contains $F_{3,1}$	101020	O_{21}
10002	contains $F_{3,2}$	102000	O_{22}
10010	O_{19}	102001	contains $F_{3,1}$
10020	O_7	102002	contains $F_{3,2}$
10100	O_{19}	102010	O_{21}
10101	contains $F_{3,1}$	102020	O_{21}
10102	contains $F_{3,2}$	110000	contains $F_{3,1}$
10110	contains $F_{3,1}$	110010	contains $F_{3,1}$
10120	O_{17}	110020	contains $F_{3,1}$
10200	O_{20}	120000	contains $F_{5,2}$
10201	contains $F_{3,1}$	120010	contains F_4
10202	contains $F_{3,2}$	120020	contains F_4

Table 4: All cycles relevant for Lemma 14.

Theorem 15. *Let C be a cycle. C is A-perfect if, and only if, C is one of the 14 configurations of Fig. 5.*

Proof. By Lemma 14, there are exactly 22 cycles with at most 6 vertices which do not contain any of the forbidden configurations $F_{3,1}$, $F_{3,2}$, F_4, resp. $F_{5,1}$ as induced subdigraphs. These cycles are those of Figs. 4 and 5. Among these cycles, there are 8 cycles (see Fig. 4) which are not A-nice, by Lemma 12. By Lemma 13, the 14 cycles of Fig. 5 are A-nice. Since their proper induced digraphs are forests of paths satisfying the condition of Theorem 9 (c), these cycles are A-perfect. By Lemma 11, cycles with more than 6 vertices are not A-perfect, therefore the list in Fig. 5 is complete. □

Corollary 16. *Let C be a cycle. C is g-perfect (resp., $[A, B]$-perfect) if, and only if, C is one of the 6 configurations O_9, O_{10}, O_{11}, O_{12}, O_{13}, or O_{17}, depicted in Fig. 5.*

Proof. The 6 configurations are $[A, B]$-nice, cf. the proof of Lemma 13, in particular the second strategy for O_{12} resp. O_{13}. The remaining 8 configurations are neither g- nor $[A, B]$-perfect, since O_{20}, O_{21} and O_{22} contain the forbidden path Q_8, O_{18} contains Q_{14}, and O_{19} conatins Q_{12}, and for O_{14}, O_{15} resp. O_{16} Bob has the following winning strategy in the g-coloring game with 2 colors. After Alice has colored a vertex with color 1, Bob colors a vertex at distance 2 with color 2. □

We now consider semiorientations of complete graphs with clique number 2. Fig. 6 depicts all semiorientations of the complete graph K_3 with clique number of at most 2. We conclude

Corollary 17. *The only A-perfect (resp. g-perfect, resp. $[A, B]$-perfect) semiorientations of K_3 with clique number of at most 2 are O_9 and O_{10}.*

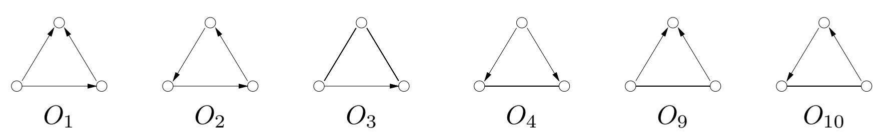


Figure 6: Semiorientations of K_3

Theorem 18. *The only A-perfect semiorientation of K_4 with clique number of at most 2 is $\overline{\vec{C}_4}$, the complement of the directed 4-cycle (see Fig. 7).*

Proof. Let D be a semiorientation of K_4 with vertices v_1, v_2, v_3, v_4. In case D has at most one edge, D contains an orientation of a K_3 (i.e. O_1 or O_2), which is not A-perfect. If D has two adjacent edges, either D contains a O_3, which is not A-perfect, or D has clique number at least 3. So we may assume that D contains the edges v_1v_2 and v_3v_4 and no further edges. W.l.o.g. the arc between v_1 and v_3 is directed as (v_1, v_3). Since the subdigraph on the vertices v_1, v_3, v_4

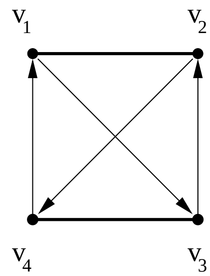


Figure 7: $\overline{\vec{C}_4}$

may not be O_4, which is not A-perfect, the arc between v_1 and v_4 is directed as (v_4, v_1). With the same arguments concerning the sets of vertices $\{v_1, v_2, v_4\}$ resp. $\{v_2, v_3, v_4\}$ one finds the orientation of the other arcs that are (v_2, v_4) resp. (v_3, v_2).

A winning strategy for Alice on $\overline{\vec{C}_4}$ is the following: She misses her first turn. Then she colors the vertex which is connected by an edge to the vertex Bob has colored. After her move the coloring is fixed and she will win. Alice also wins on every proper induced subdigraph of $\overline{\vec{C}_4}$ as follows from Theorem 17. $\square$

Theorem 19. *For $n \geq 4$, there is no semiorientation with clique number 2 of K_n that is g-perfect (resp. $[A, B]$-perfect).*

Proof. Obviously, it is sufficient to prove the theorem for $n = 4$. In view of Theorem 18, it is sufficient to describe a winning strategy for the g-coloring game with 2 colors on the digraph $\overline{\vec{C}_4}$ depicted in Fig. 7. If Alice colors a vertex v_i with the first color, then v_i has exactly one single arc of the form (v_i, v_j). Bob colors v_j with the second color. Then the vertex joined with v_i by an edge cannot be colored any more. This completes the proof. $\square$

Theorem 20. *For $n \geq 5$, there is no semiorientation with clique number 2 of K_n that is A-perfect.*

Proof. Obviously, it is sufficient to prove the theorem for $n = 5$. Let D be a semiorientation of K_5 with vertices v_1, v_2, v_3, v_4, v_5. Assume that D is A-perfect and has clique number 2. By Theorem 18, the subdigraph on the vertices v_1, v_2, v_3, v_4 must be the digraph of Fig. 7. Again, by Theorem 18 the subdigraph on the vertices v_1, v_5, v_3, v_4 must be isomorphic to the digraph of Fig. 7, in particular there must be an edge v_1v_5. But then the digraph induced by v_1, v_2, v_5 is either an undirected triangle (which contradicts the precondition that D has clique number 2) or O_3 (which is not A-perfect, again a contradiction). $\square$

6 Open problems

By a previous result [6], the class of g-perfect graphs is exactly the class of $[A, B]$-perfect graphs. In view of this, Corollary 10 and Corollary 16 suggest the following.

Conjecture 21. *For any digraph D, D is g-perfect if and only if D is $[A, B]$-perfect.*

Note that in spite of this there are digraphs D with $\chi_g(D) \neq \chi_{[A,B]}(D)$ (e.g. $P_4 \cup P_1$).

In Fig. 4, 8 forbidden cycles are depicted. These are minimal forbidden configurations, i.e. they do not contain other forbidden configurations as proper induced subdigraphs. Together with the 7 forbidden paths $F_{3,1}$, $F_{3,2}$, $F_{5,1}$, $F_{5,2}$, $F_{7,1}$, $F_{7,2}$, and F_8, and the non-connected forbidden configuration F_4, so far we have found 16 minimal forbidden configurations for A-perfectness of digraphs. There are many more minimal forbidden configurations. E.g., from the results in [4] we conclude that the chair (a tree with 5 vertices that is neither a path nor a star) is such a minimal forbidden configuration, as well as some of its semiorientations.

The next step in order to complete the list of minimal forbidden configurations for A-perfectness would be to consider forests in general, instead of forests of paths. By Lemma 6 we have that a tree of diameter $d \geq 9$ is not A-perfect. However, a lot of trees would have to be examined in order to determine the forbidden configurations. Note that the number of A-perfect trees is infinite since, for example, every in-star is A-perfect.

Conjecture 22. *The number of minimal forbidden trees is finite.*

This conjecture is in contrast to the fact that there are infinitely many minimal forbidden configurations in total, since every odd antihole (the complement of a cycle of odd length $k \geq 5$) is a minimal forbidden configuration, as remarked in [6].

Problem 23. *Give a characterization of A-perfect semiorientations of forests by a set of forbidden induced subdigraphs.*

In [4], the following theorem was proved:

Theorem 24. *A graph G with $\omega(G) \leq 2$ is A-perfect if, and only if, every component of G is either a singleton K_1 or a complete bipartite graph $K_{m,n}$ or a complete bipartite graph $K_{m,n} - e$ in which one edge e is missing (for some m, n).*

Problem 25. *Give a characterization of A-perfect digraphs with clique number 2 by a set of forbidden induced subdigraphs.*

It is clear that every component but one of an A-perfect digraph with clique number 2 must be a bipartite graph of the form as described in Theorem 24.

Problem 26. *Give a characterization of A-perfect graphs by a set of forbidden induced subgraphs.*

Some partial results for Problem 26 are given in [4, 6]. Problems 23, 25 resp. 26 are intermediate steps to solve and special cases of Problem 2. In view of our results, maybe easier to attack might be the following.

Problem 27. *Give a characterization of $[A, B]$- resp. g-perfect digraphs by a set of forbidden induced subdigraphs.*

Problem 27 has been solved in the case of undirected graphs [6].

In the weak digraph coloring game defined by Yang and Zhu [28] the players may color a vertex in a color that does not create a monochromatic directed cycle. For weak S-game chromatic numbers we might also define weak S-perfectness of a digraph. The analog of Problem 2 is the following.

Problem 28. *For $S \in \mathcal{S}$, characterize the class of weakly S-perfect digraphs.*

Note that in the case of (undirected) graphs weak S-perfectness and S-perfectness coincide. We remark that the classes of weakly S-perfect digraphs are richer than the classes of S-perfect digraphs, e.g. the configurations $F_{3,1}$, $F_{3,2}$, F_4, $F_{5,1}$, $F_{7,1}$, $F_{7,2}$, and F_8 are weakly A-perfect (but not A-perfect).

Conjecture 29. *For any $S \in \mathcal{S}$ and any digraph D, if D is S-perfect, then D is weakly S-perfect.*

References

[1] S.D. Andres, The game chromatic index of forests of maximum degree $\Delta \geq 5$, Discrete Applied Math. **154** (2006) 1317–1323

[2] S.D. Andres, Lightness of digraphs in surfaces and directed game chromatic number, Discrete Math. **309** (2009) 3564–3579

[3] S.D. Andres, Asymmetric directed graph coloring games, Discrete Math **309** (2009) 5799–5802

[4] S.D. Andres, Game-perfect graphs, Mathematical Methods of Operations Research **69** (2009) 235–250

[5] S.D. Andres, Directed defective asymmetric graph coloring games, Discrete Applied Math. **158** (2010) 251–260

[6] S.D. Andres, On characterizing game-perfect graphs by forbidden induced subgraphs, to appear in: Contributions to Discrete Math.

[7] S.D. Andres, W. Hochstättler, A strong perfect digraph theorem, submitted for publication

[8] J. Bang-Jensen, G. Gutin, Digraphs. Theory, algorithms and applications, Springer-Verlag London Ltd, London, 2009

[9] T. Bartnicki, B. Brešar, J. Grytczuk, M. Kovše, Z. Miechowicz, I. Peterin, Game chromatic number of Cartesian product graphs, Electronic J. Comb. **15** (2008), R72

[10] C. Berge, Graphs and hypergraphs, North Holland Publishing Company, Amsterdam, London, 1976

[11] H.L. Bodlaender, On the complexity of some coloring games, Int. J. Found. Comput. Sci. **2**, no.2 (1991) 133–147

[12] T. Bohman, A. Frieze, B. Sudakov, The game chromatic number of random graphs, Random Struct. Algorithms **32** (2008), 223–235

[13] D. Bokal, G. Fijavž, M. Juvan, P.M. Kayll, B. Mohar, The circular chromatic number of a digraph, J. Graph Theory **46** (2004) 227–240

[14] L. Cai, X. Zhu, Game chromatic index of k-degenerate graphs, J. Graph Theory **36** (2001) 144–155

[15] M. Chudnovsky, N. Robertson, P. Seymour, R. Thomas, The strong perfect graph theorem, Ann. Math. **164** (2006) 51–229

[16] P. Erdös, U. Faigle, W. Hochstättler, W. Kern, Note on the game chromatic index of trees, Theoretical Comp. Sci. **313** (2004) 371–376

[17] L. Esperet, X. Zhu, Game colouring the square of graphs, Discrete Math. **309** (2009), 4514–4521

[18] U. Faigle, W. Kern, H. Kierstead, W.T. Trotter, On the game chromatic number of some classes of graphs, Ars Combin. **35** (1993) 143–150

[19] W. He, X. Hou, K.-W. Lih, J. Shao, W. Wang, X. Zhu, Edge-partitions of planar graphs and their game coloring numbers, J. Graph Theory **41**, no.4 (2002) 307–317

[20] H. Jacob, H. Meyniel, Extension of Turan's and Brook's Theorems and new notions of stability and coloring in digraphs, Ann. Discrete Math. **17** (1983) 365–370

[21] K. Junosza-Szaniawski, L. Rożej, Game chromatic number of graphs with locally bounded number of cycles, Inform. Process. Lett. **110** (2010), 757–760

[22] H.A. Kierstead, A simple competitive graph coloring algorithm, J. Comb. Theory **B 78** (2000) 57–68

[23] L. Lovász, Normal hypergraphs and the perfect graph conjecture, Discrete Math. **2** (1972) 253–267

[24] J. Nešetřil, E. Sopena, On the oriented game chromatic number, Electronic J. Combinatorics **8**, no.2 (2001) R14

[25] V. Neumann-Lara, The dichromatic number of a digraph, J. Combin. Theory B 33 (1982) 265–270

[26] E. Sidorowicz, The game chromatic number and the game colouring number of cactuses, Inform. Process. Lett. **102** (2007), 147–151

[27] J. Wu, X. Zhu, Lower bounds for the game colouring number of partial k-trees and planar graphs, Discrete Math. **308** (2008), 2637–2642

[28] D. Yang, X. Zhu, Game colouring directed graphs, Electron. J. Combin. **17** (2010) R11

[29] X. Zhu, The game coloring number of pseudo partial k-trees, Discrete Math. **215** (2000) 245–262

[30] X. Zhu, Refined activation strategy for the marking game, J. Comb. Theory **B 98** (2008) 1–18