# Diskrete Mathematik und Optimierung

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**Topological Sweeping in Oriented Matroids**

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# Topological Sweeping in Oriented Matroids

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**Abstract**

According to [1] 10.4.6 it is in general impossible to sweep an oriented matroid of rank $\geq 4$ with a hyperplane. The authors immediately afterwards remark that

> topological sweeping can be generalized to "Euclidean pseudoarrangements".

This is cited and used in [2] but seems not to have been made explicit anywhere. The purpose of this note is to fill that void.

## 1 Topological Sweeping in Oriented Matroids

**Definition 1** ([1] 10.1.3, 10.5.2)**.** *1. An* oriented matroid program *is a triple* $(\mathcal{O}, g, f)$*, where* $\mathcal{O}$ *is an oriented matroid on a finite set* $E$*,* $g, f \in E$ *where* $g$ *is not a loop,* $f$ *is not a coloop and* $g \neq f$*. We call* $g$ *the* hyperplane at infinity *and* $f$ *the* objective function *or* sweeping plane*.*

2. *A single element extension* $\tilde{\mathcal{O}} = \mathcal{O} \cup h$*, is such that* $h$ is parallel to $f$ with respect to $g$*, if for all* $Y \in \mathcal{O}$ *we have* $Y_g = 0 \Rightarrow Y_f = Y_h$*, or, in other words, if* $h$ *and* $f$ *and* $h$ *are parallel in* $\tilde{\mathcal{O}}/g$*.*

3. *An* oriented matroid program $(\mathcal{O}, g, f)$ *is* Euclidean *if for every cocircuit* $Y^0 \in \mathcal{O}$ *such that* $Y^0_g = +$*, there exists a single element extension* $\tilde{\mathcal{O}} = \mathcal{O} \cup h$*, such that* $h$ *is parallel to* $f$ *with respect to* $g$*, and* $(Y^0, 0)$ *is a cocircuit of* $\tilde{\mathcal{O}}$*.*

From now on we will, furthermore, tacitly assume, that $g$ is not a coloop, $f$ is not a loop and $f$ and $g$ are not parallel.

Next we consider an orientation of (some of) the edges of the 1-skeleton of the affine oriented matroid $(\mathcal{O}, g)$.

**Definition 2** (see [1] 10.1.16)**.** *Let $G_f$ be the graph whose vertices are the cocircuits of $(\mathcal{O}, g)$ and whose edges are the covectors of rank 2 in $\mathcal{O}$ between them. If $Y^1, Y^2$ are compatible vertices such that $Y^1 \circ Y^2$ has rank 2, i.e. $(Y^1, Y^2)$ is an edge in $G_f$, let $Z$ denote the cocircuit derived by corcircuit elimination of $g$ between $Y^2$ and $-Y^1$. If $Z_f = +$ we direct $(Y^1, Y^2)$ from $Y^1$to $Y^2$, if $Z_f = -$ we direct $(Y^1, Y^2)$ from $Y^2$to $Y^1$. In the (degenerate) case that $Z_f = 0$ the edge $(Y^1, Y^2)$ remains undirected.*

*A* directed cycle *in $G_f$ with that orientation is a cycle which contains at least one directed edge and which has a traversal where all directed edges are traversed in forward direction.*

In Figure 1 we depicted an oriented matroid program with the direction of its edges and an extension by an element $h$ parallel to $f$ wrt. $g$.

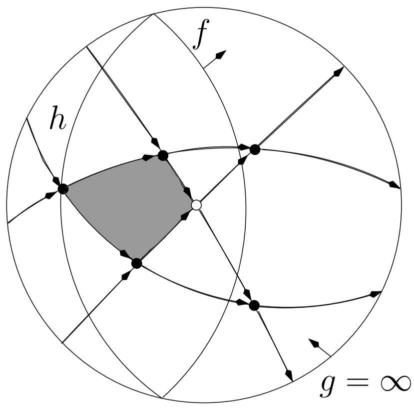


Figure 1: An oriented matroid program in rank 3

The undirected edges can be considered as directions in $f$. To make this more precise we need the notion of a linear subclass of a matroid. In the following we denote by $M(\mathcal{O})$ the matroid underlying the oriented matroid $\mathcal{O}$.

**Definition 3** (see [3], 7.2 Exercise 6)**.** *A* linear subclass *of a matroid $M$ is a subset $\mathcal{H}'$ of its set of hyperplanes which with any modular pair $H_1, H_2 \in \mathcal{H}'$ must contain all hyperplanes that cover $H_1 \cap H_2$.*

It is well known ([3], 7.2 Exercise 6 (iii), 7.2.1) that the non-trivial linear subclasses of a matroid are in bijection with its non-trivial one-point-extensions.

**Proposition 4.** *Let $(\mathcal{O}, g, f)$ denote an oriented matroid program and $G_f$ its graph. Then $(X, Y)$ is an undirected edge in $G$ if and only if $z(Y)$ is a* *member of the smallest linear subclass $\mathcal{H}$ containing $z(X)$ and all hyperplanes containing $g$ and $f$ in $M(\mathcal{O})$, where $z(X)$ denotes the zero set of $X$.*

*Proof.* $(X, Y)$ is an undirected edge in $G_f$, if and only if $f$ is contained in $H_3 = (z(X) \wedge z(Y)) \vee g$. Since $H_3$ is a hyperplane in $M(\mathcal{O})$ which contains $f$ and $g$, and $H_3$ and $z(X)$ form a modular pair, $z(Y)$ must be a member of $\mathcal{H}$. On the other hand all colines covered by at least two hyperplanes in the smallest linear subclass $\mathcal{H}'$ containing $z(X)$ and $g$ and $f$ must be covered by a hyperplane containing $g$ and $f$. If this were not the case, then $f \vee g$ and such a coline $l$ would form a modular pair, implying that the modular cut associated with $\mathcal{H}'$ (see [3], 7.2 Exercise 6 (iii)) is the full geometric lattice and thus, $h$ must be a loop contradicting $(\mathcal{O}, g, f)$ being Euclidean. Hence, if $(X, Y)$ is an edge and $z(Y)$ is contained in $\mathcal{H}'$, then $z(X) \wedge z(Y)$ must be covered by a hyperplane containing $f$ and $g$ and thus $(X, Y)$ must be undirected. $\square$

Hence every extension through a cocircuit $X$ parallel to $f$ wrt. $g$ goes exactly through the cocircuits reachable from $X$ on a path using only undirected edges.

We have the following theorem linking Euclidean oriented matroid programs and $G_f$

**Theorem 5** (Edmonds-Mandel, see [1] 10.5.5, 10.5.10)**.** *Let $(\mathcal{O}, g, f)$ denote an oriented matroid program, and $Y^0 \in (\mathcal{O}, g)$ be a cocircuit. Then there is a Euclidean extension of $(\mathcal{O}, g, f)$ by a hyperplane $h$ through $Y^0$ parallel to $f$ wrt. $g$, if and only if $Y^0$ is not contained in a directed cycle of $G_f$.*

*In particular, $(\mathcal{O}, g, f)$ is Euclidean if and only if $G_f$ contains no directed cycle.*

The restrictions on the signs of the old vertices wrt. the new element are given by:

**Lemma 6** (see [1] 10.5.3)**.** *Let $(\mathcal{O}, g, f)$ be an oriented matroid program, $Y^0 \in (\mathcal{O}, g)$ a cocircuit, and $(\tilde{\mathcal{O}} = \mathcal{O} \cup h, g, f)$ be an extension through $Y^0$ by a hyperplane parallel to $f$ wrt. $g$. Let $Y_1, Y_2 \in (\mathcal{O}, g)$ be two conformal cocircuits such that $Y_1 \circ Y_2$ is an edge. Let $\tilde{Y}^1, \tilde{Y}^2$ denote the corresponding cocircuits in $\tilde{\mathcal{O}}$. Then, if $(Y_1, Y_2)$ is undirected we have $\tilde{Y}^1_h = \tilde{Y}^2_h$ and if $(Y_1, Y_2)$ is directed from $Y_1$ to $Y_2$, we have $\tilde{Y}^1_h \preceq \tilde{Y}^2_h$ with respect to the order $- \prec 0 \prec +$, where $\tilde{Y}^1_h = \tilde{Y}^2_h = 0$ cannot occur.*

We observe, that hyperplanes parallel wrt. $g$ cannot cross in the affine oriented matroid:

**Proposition 7.** *Let $(\mathcal{O}, g)$ be an affine oriented matroid and $(\tilde{\mathcal{O}}, g) = (\mathcal{O} \cup \{f_1, f_2\}, g)$ an extension by two elements that are parallel wrt. $g$. Then there do not exist $X, Y$ in $\tilde{\mathcal{O}}$ such that $X_g = Y_g = +, X_{f_1} = Y_{f_2} = +$ and $X_{f_2} = Y_{f_1} = -$.*

*Proof.* Assume there were two such elements. Then covector elimination of $g$ between $X$ and $-Y$ would yield a vector $Z$ satisfying $Z_g = 0, Z_{f_1} = +$ and $Z_{f_2} = -$ contradicting $f_1, f_2$ being parallel wrt. $g$. □

After these preparations we can show the possibility of topological sweeping in Euclidean oriented matroid programs.

**Definition 8.** *Let $(\mathcal{O}, f, g)$ be a Euclidean oriented matroid program and let $\mathcal{D}_+$ denote its set of* affine cocircuits, *i.e. those cocircuits $Y \in \mathcal{O}$ satisfying $Y_g = +$.*

*A* topological sweep *of $(\mathcal{O}, f, g)$ is a total order $Y^1, \ldots, Y^r$ of $\mathcal{D}_+$, the* sweep order, *together with an $r$-point-extension $\tilde{\mathcal{O}} = \mathcal{O} \cup \{h_1, \ldots, h_r\}$ such that*

1. *forall $1 \leq i \leq r$ $h_i$ is parallel to $f$ wrt. $g$*
2. *forall $1 \leq i \leq r$ there exist $1 \leq i_1 \leq i \leq i_2$ such that*

$$\tilde{Y}^i_{h_j} = \begin{cases} - & \text{if} \quad j < i_1 \\ 0 & \text{if} \quad i_1 \leq j \leq i_2 \\ + & \text{if} \quad j > i_2. \end{cases}$$

*We say that the sweep is non-degenerate if $i_1 = i = i_2$ for all $1 \leq i \leq r$.*

**Theorem 9.** *Let $(\mathcal{O}, g, f)$ be a Euclidean oriented matroid program. Then there exists a topological sweep of $\mathcal{O}$. The sweep is non-degenerate if and only $f$ and $g$ are not contained in a common hyperplane $h$ of $M(\mathcal{O})$ that covers a coline $l$ disjoint from $f$ and $g$.*

*Proof.* Let $Y^1, \ldots, Y^r$ denote the affine cocircuits of $(\mathcal{O}, g)$. The undirected edges in $G_f$ define equivalence classes. Pick one representative from each class and extend the program by an element parallel to $f$ wrt. $g$ for each such representative, this way generating extensions $h_1, \ldots, h_k$. Then by Lemma 6 and Proposition 4 for every vertex $\tilde{Y}^i$ of the extension there exists a unique $j$ such that $\tilde{Y}^i_{h_j} = 0$. Now, if $\tilde{Y}^{i_1}, \tilde{Y}^{i_2}$ are two affine cocircuits such that $\tilde{Y}^{i_1}_{h_{j_1}} = 0$ and $\tilde{Y}^{i_2}_{h_{j_2}} = 0$ where $j_1 \neq j_2$, then by Proposition 7 the sets of indices from $\{h_1, \ldots, h_k\} \setminus \{h_{j_1}, h_{j_2}\}$ with positive signs on $Y^{i_1}$ and $Y^{i_2}$ must not cross, i.e. one is contained in the other. By Lemma 6 the containment must be strict unless one of $\tilde{Y}^{i_1}_{h_{j_2}}$ and $\tilde{Y}^{i_2}_{h_{j_1}}$ is positive and the other one negative. This induces a linear order on the equivalence classes of cocircuits. This linear order finally yields the topological sweep if we add sufficiently many parallel copies of an extension through an equivalence class containing more than one vertex. $\square$

**Remark 10.** *We can and frequently will identify a single element extension* $\tilde{\mathcal{O}} = \mathcal{O} \cup h$ *with its localization* $\sigma_h$ *in* $\mathcal{O}$ *where for all* $Y \in \mathcal{O} : \sigma_h(Y) = \varepsilon$ *if and only if* $(Y, \varepsilon) \in \tilde{\mathcal{O}}$.

# References


[1] Anders Björner, Michel Las Vergnas, Bernd Sturmfels, Neil White, and Günter M. Ziegler, *Oriented matroids*, Cambridge University Press, March 1993.

[2] Emanuele Delucchi and Simona Settepanella, *Combinatorial polar orderings and recursively orderable arrangements*, Advances in Applied Mathematics **44** (2010), no. 2, 124 – 144.

[3] James G. Oxley, *Matroid Theory*, Oxford Science Publications, 1992.