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ON FINDING SOME NEW EXCLUDED MINORS FOR GAMMOIDS

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Abstract. The class of gammoids is the smallest class of matroids closed under duality and minors that contains the class of transversal matroids. Ingleton showed that the class of gammoids does not allow for a finite characterization in terms of excluded minors [4], and Mayhew showed that every gammoid may be extended to a matroid that is an excluded minor for the class of gammoids [7]. The properties of the antichain of excluded minors has yet to be examined thoroughly. The natural approach to this would be to start with some non-gammoid and then examine its minors for the minimal minors that are not gammoids. Although it is comparatively easy to check whether a given matroid is either a strict gammoid or a transversal matroid, we still have to deal with minors that are neither strict gammoids nor transversal matroids. We introduce an easily implemented and automated sufficient condition that a given matroid is not a gammoid, which greatly reduces the handwork needed in this investigation.

1. Preliminaries

Mason introduced the following notion of a gammoid and a strict gammoid in [6].

Definition 1. *Let $D = (V, A)$ be a digraph, $E \subseteq V$, and $S \subseteq V$ be a set of vertices that are called* **sinks**. *The* **gammoid represented by** (D, S, E) *has the ground set E – which elements we call* **matroid elements** *– and any set $X \subseteq E$ is independent, if there is a family $\mathcal{P}$ of pair-wise vertex disjoint paths in D, such that every path $p \in \mathcal{P}$ ends in a sink vertex $p_s \in S$, and such that for every $x \in X$, there is a path $p_x \in \mathcal{P}$ that starts in the vertex x.*

A matroid M is called **gammoid**, *if it is isomorphic to a gammoid represented by (D, S, E) for some digraph $D = (V, A)$, $E \subseteq V$, and $S \subseteq V$.*

A matroid M is called **strict gammoid**, *if it is isomorphic to a gammoid represented by (D, S, V) for some digraph $D = (V, A)$ and $S \subseteq V$.*

Lemma 2. *Let M be a matroid on the ground set E.*

M is a gammoid if and only if there is a strict gammoid N on the ground set $\bar{E} \supseteq E$, such that for all $X \subseteq E$, $\mathrm{rk}_N(X) = \mathrm{rk}_M(X)$.

Proof. Assume that M is a gammoid, then there is some (D, S, E) with $D = (V, A)$ such that M is isomorphic to the gammoid represented by (D, S, E). The desired N is defined on $\bar{E} = V$ and it is isomorphic to the strict gammoid represented by (D, S, V). □

Ingleton and Piff proved in [5] that the strict gammoids are exactly the co-transversal matroids. Bonin, Kung, and De Mier observed in [2]:

Theorem 3. *Let M be a matroid, and let $\mathcal{Z}(M)$ be the set of all cyclic flats of M. M is a strict gammoid if and only if for all $\mathcal{F} \subseteq \mathcal{Z}(M)$ the inequality*

$$\mathrm{rk}(\cup\mathcal{F}) \leq \sum_{\emptyset\neq\mathcal{F}'\subseteq\mathcal{F}} (-1)^{|\mathcal{F}'|+1}\,\mathrm{rk}(\cap\mathcal{F}')$$

holds.

In [3], Crapo established the following one-to-one correspondence between single element extensions of a matroid and modular cuts in the lattice of flats of that matroid.

Definition 4. *Let $\mathcal{P} = (P, \leq)$ be a lattice, $p, q \in P$, $p \leq q$ and $p \neq q$.*
*We say that q **covers** p, if for all $p' \in P$ with $p \leq p' \leq q$, $p' \in \{p, q\}$.*
*Let $F \subseteq P$. F is a **cut** of $\mathcal{P}$, if for all $f \in F$ and all $p \in P$ the implication*

$$f \leq p \Rightarrow p \in F$$

holds.
*A cut F is called **modular**, if for every $f, g \in F$ the implication*

$$f \text{ covers } f \wedge g \Rightarrow f \wedge g \in F$$

holds.

Note that the last implication is equivalent to requiring that if $A, B \in F$ form a modular pair, i.e. $\mathrm{rk}(A) + \mathrm{rk}(B) = \mathrm{rk}(A \cap B) + \mathrm{rk}(A \cup B)$, then also $A \cap B \in F$.

Theorem 5. *Let M be a matroid on E, $\mathcal{F}(M)$ be the set of all flats of M, $e \notin E$. If N is a matroid on $E \cup \{e\}$, such that for all $X \subseteq E$, $\mathrm{rk}_N(X) = \mathrm{rk}_M(X)$, then*

$$\{F \in \mathcal{F}(M) \mid \mathrm{rk}_M(F) = \mathrm{rk}_N(F \cup \{e\})\}$$

is a modular cut of the lattice $(\mathcal{F}(M), \subseteq)$.
Conversely, if $\mathcal{J} \subseteq \mathcal{F}(M)$ is a modular cut of the lattice $(\mathcal{F}, \subseteq)$, then

$$\mathrm{rk}_{M+\mathcal{J}} \colon 2^{E\cup\{e\}} \longrightarrow \mathbb{N},\ X \mapsto \mathrm{rk}_M(X \cap E) + \begin{cases} 1 & \text{if } e \in X \text{ and } \mathrm{cl}_M(X \cap E) \notin \mathcal{J} \\ 0 & \text{otherwise} \end{cases}$$

is the rank-function of a single-element extension of M.

The proof is in [3].

2. Single-Element Extensions And Slack

Definition 6. *Let M be a matroid, $\mathcal{Z}(M)$ the set of all cyclic flats of M, $\mathcal{F}(M)$ the set of all flats of M. The map*

$$s \colon 2^{\mathcal{Z}(M)} \longrightarrow \mathbb{Z},\ \mathcal{A} \mapsto -\mathrm{rk}(\cup\mathcal{A}) + \sum_{\emptyset\neq\mathcal{A}'\subseteq\mathcal{A}} (-1)^{|\mathcal{A}'|+1}\,\mathrm{rk}(\cap\mathcal{A}')$$

*is called **slack map** of M.*
*Let $\mathcal{A} \subseteq \mathcal{Z}(M)$. $\mathcal{A}$ is called **violation**, if $s(\mathcal{A}) < 0$.*

Observe that $\mathcal{A} \subseteq \mathcal{Z}(M)$ is a violation, if and only if the inequality for strict gammoids presented in Theorem 3 does not hold for $\mathcal{A}$. Therefore, a matroid is a strict gammoid if and only if there is no violation $\mathcal{A} \subseteq \mathcal{Z}(M)$.

Definition 7. *Let M be a matroid, $\mathcal{Z}(M)$ be the set of all cyclic flats of M, and $\mathcal{J}$ be a modular cut of the lattice of flats $(\mathcal{F}(M), \subseteq)$.*

We define the **slack difference** *of $\mathcal{J}$ to be the map*

$$\delta_{\mathcal{J}} \colon 2^{\mathcal{Z}(M)} \longrightarrow \mathbb{N},\ \mathcal{X} \mapsto \begin{cases} 0 & \mathcal{X} = \emptyset \text{ or } \mathcal{X} \not\subseteq \mathcal{J} \text{ or } \cap \mathcal{X} \in \mathcal{J} \\ 1 & \text{otherwise, in this case } \emptyset \neq \mathcal{X} \subseteq \mathcal{J},\ \cap \mathcal{X} \notin \mathcal{J}. \end{cases}$$

Lemma 8. *Let M be a matroid, $\emptyset \neq \mathcal{A} \subseteq \mathcal{Z}(M)$ a non-empty family of cyclic flats, $\mathcal{J}$ be a modular cut of $(\mathcal{F}(M), \subseteq)$. Let N be the single-element extension of M that corresponds to $\mathcal{J}$, and let $e \in \mathrm{E}(N)\backslash\mathrm{E}(M)$ denote the extended element in N. Then:*

(i) *The map $\eta \colon \mathcal{A} \xrightarrow{\sim} \{\mathrm{cl}_N(A) \mid A \in \mathcal{A}\}$, $X \mapsto \mathrm{cl}_N(X)$ is a bijection.*
(ii) $\forall \mathcal{A}' \subseteq \mathcal{A} \colon\ \mathrm{rk}_N(\cap\{\mathrm{cl}_N(A) \mid A \in \mathcal{A}\}) = \mathrm{rk}_M(\cap\mathcal{A}) + \delta_{\mathcal{J}}(\mathcal{A})$.
(iii) $s_N(\{\mathrm{cl}_N(A) \mid A \in \mathcal{A}\}) = s_M(\mathcal{A}) + \sum_{\emptyset \neq \mathcal{A}' \subseteq \mathcal{A}} (-1)^{|\mathcal{A}'|+1} \delta_{\mathcal{J}}(\mathcal{A}')$

Proof. Let $\mathcal{B} = \{\mathrm{cl}_N(A) \mid A \in \mathcal{A}\}$. Since $\mathcal{A}$ is finite, it is sufficient to show that η is injective. Let $A, A' \in \mathcal{A}$ such that $\eta(A) = \eta(A')$. Since A and A' are both flats in M, we get that $\eta(A)\backslash A \subseteq \{e\}$ and $\eta(A')\backslash A' \subseteq \{e\}$. Therefore $\eta(A) = \eta(A')$ implies that $A \cap \mathrm{E}(M) = A' \cap \mathrm{E}(M)$. But since $A, A' \subseteq \mathrm{E}(M)$, we obtain that $A = A'$, so η is injective, *(i)* holds.

Now let $\emptyset \neq \mathcal{A}' \subseteq \mathcal{A}$ be arbitrary and fixed, and let $\mathcal{B}' = \{\eta(A) \mid A \in \mathcal{A}'\}$. If $\mathcal{A}' \not\subseteq \mathcal{J}$, then there is some $A \in \mathcal{A}'\backslash\mathcal{J}$ with $A = \mathrm{cl}_N(A)$. Thus $e \notin \cap\mathcal{B}'$, therefore $\cap\mathcal{B}' = \cap\mathcal{A}'$. Since N is an extension of M, we have that $\mathrm{rk}_M(\cap\mathcal{A}') = \mathrm{rk}_N(\cap\mathcal{B}')$. If $\mathcal{A}' \subseteq \mathcal{J}$, then $e \in \cap\mathcal{B}'$. Thus $\mathrm{rk}_N(\cap\mathcal{B}') = \mathrm{rk}_N(\cap\mathcal{A}') = \mathrm{rk}_M(\cap\mathcal{A}')$ if $\cap\mathcal{A}' \in \mathcal{J}$. Otherwise $\mathrm{rk}_N(\cap\mathcal{B}') = \mathrm{rk}_M(\cap\mathcal{A}') + 1$ holds. This is in perfect alignment with the definition of $\delta_{\mathcal{J}}$, so we obtain the equation *(ii)*

$$\mathrm{rk}_N(\cap\mathcal{B}') = \mathrm{rk}_M(\cap\mathcal{A}') + \delta_{\mathcal{J}}(\mathcal{A}').$$

Analogously, we obtain that $\mathrm{rk}_N(\cup\mathcal{B}) = \mathrm{rk}_M(\cup\mathcal{A})$: if $\mathcal{A} \cap \mathcal{J} = \emptyset$, then $\cup\mathcal{B} = \cup\mathcal{A}$, otherwise $\mathrm{cl}_M(\cup\mathcal{A}) \in \mathcal{J}$ since $\mathcal{J}$ is closed under supersets.

Let us rewrite the definition of $s_N(\mathcal{B})$ using the above equalities. We obtain

$$\begin{aligned} s_N(\mathcal{B}) &= -\mathrm{rk}_N(\cup\mathcal{B}) + \sum_{\emptyset \neq \mathcal{B}' \subseteq \mathcal{B}} (-1)^{|\mathcal{B}'|+1} \mathrm{rk}_N(\cap\mathcal{B}') \\ &= -\mathrm{rk}_M(\cup\mathcal{A}) + \sum_{\emptyset \neq \mathcal{A}' \subseteq \mathcal{A}} (-1)^{|\mathcal{A}'|+1} \mathrm{rk}_N(\cap\{\eta(A) \mid A \in \mathcal{A}'\}) \\ &= -\mathrm{rk}_M(\cup\mathcal{A}) + \sum_{\emptyset \neq \mathcal{A}' \subseteq \mathcal{A}} (-1)^{|\mathcal{A}'|+1} \left(\mathrm{rk}_M(\cap\mathcal{A}') + \delta_{\mathcal{J}}(\mathcal{A}')\right) \\ &= s_M(\mathcal{A}) + \sum_{\emptyset \neq \mathcal{A}' \subseteq \mathcal{A}} (-1)^{|\mathcal{A}'|+1} \delta_{\mathcal{J}}(\mathcal{A}'). \end{aligned}$$

□

3. Stuck Families of Cyclic Flats

Definition 9. *Let M be a matroid, $\mathcal{A} \subseteq \mathcal{Z}(M)$ a family of cyclic flats, and $k \in \mathbb{N}$. We define the set of* **stuck flats** *of degree k with respect to $\mathcal{A}$ inductively:*

$$\begin{aligned} S_1(\mathcal{A}) &= \mathcal{A} \\ S_{k+1}(\mathcal{A}) &= S_k \cup \{P \cap Q \mid P, Q \in S_k,\ P \text{ covers } P \cap Q \text{ in } \mathcal{F}(M)\}. \end{aligned}$$

Lemma 10. *Let M be a matroid, $\mathcal{A} \subseteq \mathcal{Z}(M)$ a family of cyclic flats and $k = |\mathcal{A}|$. Then $S_{k+1}(\mathcal{A}) = S_k(\mathcal{A})$.*

Proof. By induction on k. If $k \leq 1$, then clearly $S_i(\mathcal{A}) = S_1(\mathcal{A}) = \mathcal{A}$ for all $i \in \mathbb{N}$. Now, let $A \in \mathcal{A}$, then

$$S_{|\mathcal{A}'|}(\mathcal{A}') = S_{|\mathcal{A}'|+1}(\mathcal{A}') = \cdots = S_{k-1}(\mathcal{A}') = S_k(\mathcal{A}') = S_{k+1}(\mathcal{A}')$$

where $\mathcal{A}' = \mathcal{A}\backslash \{A\} \subsetneq \mathcal{A}$ by induction hypothesis. Consider $F \in S_k(\mathcal{A})\backslash S_{k-1}(\mathcal{A})$, then there are $P, Q \in S_{k-1}(\mathcal{A})$ such that $F = P \cap Q$, yet for all $A \in \mathcal{A}$, $\{P, Q\} \nsubseteq S_{k-1}(\mathcal{A}\backslash \{A\})$, because otherwise $F \in S_k(\mathcal{A}\backslash \{A\})$ and thus $F \in S_{k-1}(\mathcal{A}\backslash \{A\})$ would be a contradiction. But then, for all $A \in \mathcal{A}$, either $P \subseteq A$ or $Q \subseteq A$, which implies that $F \subseteq \cap\mathcal{A}$. Assume there is some $X \in S_{k+1}(\mathcal{A})\backslash S_k(\mathcal{A})$, then $X = P \cap Q$ for $P, Q \in S_k(\mathcal{A})$ with the property that $\{P, Q\} \nsubseteq S_k(\mathcal{A})$. W.l.o.g. $P \notin S_{k-1}(\mathcal{A})$ and therefore $P \subseteq \cap\mathcal{A}$. On the other hand, every element $Q \in S_k(\mathcal{A})$ may be written as $Q = \cap\mathcal{A}'$ for some $\mathcal{A}' \subseteq \mathcal{A}$. Therefore, $X = P \cap Q = P \cap (\cap\mathcal{A}') = P$ since $P \subseteq \cap\mathcal{A} \subseteq \cap\mathcal{A}'$, contradicting that $X \notin S_k(\mathcal{A})$. □

Lemma 11. *Let M be a matroid, $\mathcal{A} \subseteq \mathcal{Z}(M)$ a family of cyclic flats, and let $\emptyset \neq \mathcal{A}' \subseteq \mathcal{A}$ be a non-empty subfamily.*

$$\cap\mathcal{A}' \in S_{|\mathcal{A}|}(\mathcal{A}) \quad \Longrightarrow \quad \cap\mathcal{A}' \in S_{|\mathcal{A}'|}(\mathcal{A}').$$

Proof. By induction on $|\mathcal{A}'|$. $\mathcal{A}' = \{A\}$ clearly has the property:

$$\cap\mathcal{A}' = A \in S_1(\mathcal{A}') \subseteq S_{|\mathcal{A}'|}(\mathcal{A}').$$

In the general case $\cap\mathcal{A}' \in S_{|\mathcal{A}|}(\mathcal{A})$ implies that there are two proper subfamilies $\mathcal{B}, \mathcal{B}'$ of $\mathcal{A}'$ such that $\cap\mathcal{B}, \cap\mathcal{B}' \in S_{|\mathcal{A}|-1}(\mathcal{A})$, $\cap\mathcal{A}'$ is covered by $\cap\mathcal{B}$, and where $\cap\mathcal{A}' = (\cap\mathcal{B}) \cap (\cap\mathcal{B}')$. By induction hypothesis, $\cap\mathcal{B} \in S_{|\mathcal{B}|}(\mathcal{B})$ and $\cap\mathcal{B}' \in S_{|\mathcal{B}'|}(\mathcal{B}')$. Thus $\cap\mathcal{A}' \in S_{\max\{|\mathcal{B}|, |\mathcal{B}'|\}+1}(\mathcal{A}') \subseteq S_{|\mathcal{A}'|}(\mathcal{A}')$. □

Definition 12. *Let M be a matroid, $\mathcal{A} \subseteq \mathcal{Z}(M)$ be a family of cyclic flats. We say that $\mathcal{A}$ is* **stuck** *in $\mathcal{F}(M)$, if $\cap\mathcal{A} \in S_{|\mathcal{A}|}(\mathcal{A})$.*

Lemma 11 immediately has the consequence that every stuck $\mathcal{A}$ can be decomposed into a strict chain $\mathcal{A}_1 \subsetneq \mathcal{A}_2 \subsetneq \cdots \subsetneq \mathcal{A}_k = \mathcal{A}$ where $k = |\mathcal{A}|$, and where each $\mathcal{A}_i$ is stuck in $\mathcal{F}(M)$, too. Also note that although $\mathcal{A}_i \subsetneq \mathcal{A}_{i+1}$, $\cap\mathcal{A}_i \subseteq \cap\mathcal{A}_{i+1}$ is not necessarily strict, but if it is strict, then the left-hand flat is a cover of the right-hand flat.

Lemma 13. *Let M be a matroid, $\mathcal{A} \subseteq \mathcal{Z}(M)$ be a family of cyclic flats, and N be an extension of M by the element $e \notin \mathrm{E}(M)$ corresponding to the modular cut $\mathcal{J} \subseteq \mathcal{F}(M)$. If $\mathcal{A}$ is stuck in $\mathcal{F}(M)$, then*

$$\mathrm{rk}_M(\cap\mathcal{A}) = \mathrm{rk}_N\left(\cap\{\mathrm{cl}_N(A) \mid A \in \mathcal{A}\}\right).$$

Proof. If $\mathcal{A} \nsubseteq \mathcal{J}$, then there is some $A \in \mathcal{A}$ such that $\mathrm{cl}_N(A) = A$, and therefore $\cap\{\mathrm{cl}_N(A) \mid A \in \mathcal{A}\} = \cap\mathcal{A}$ must have same rank in the extension N as it has in M. Otherwise, $\mathcal{A} \subseteq \mathcal{J}$. We show inductively, that for all $i \in \{1, \ldots, |\mathcal{A}|\}$ and all $F \in S_i(\mathcal{A})$, we have $e \in \mathrm{cl}_N(F)$. This is obvious for $F \in S_1(\mathcal{A}) = \mathcal{A}$. Now let $F \in S_i(\mathcal{A})$, then there are $P, Q \in S_{i-1}(\mathcal{A})$, such that $F = P \cap Q$, and such that P covers F. By induction hypothesis, $\mathrm{cl}_N(P) = P \cup \{e\}$ as well as $\mathrm{cl}_N(Q) = Q \cup \{e\}$, therefore $P, Q \in \mathcal{J}$. But since $\mathcal{J}$ is a modular cut, this implies that $F \in \mathcal{J}$, which in turn gives $\mathrm{cl}_N(F) = F \cup \{e\}$. Therefore $\mathrm{rk}_N(F \cup \{e\}) = \mathrm{rk}_N(F) = \mathrm{rk}_M(F)$. Since $\mathcal{A}$ is stuck, we obtain the desired equation from $\cap\mathcal{A} \in S_{|\mathcal{A}|}(\mathcal{A})$ and the fact that

$$\cap\{\mathrm{cl}_N(A) \mid A \in \mathcal{A}\} = \cap\{A \cup \{e\} \mid A \in \mathcal{A}\} = \cap\mathcal{A} \cup \{e\}.$$

□

Lemma 14. *Let M be a matroid, $\mathcal{A} \subseteq \mathcal{Z}(M)$ be a family of cyclic flats, and N be an extension of M by the element $e \notin \mathrm{E}(M)$ corresponding to the modular cut $\mathcal{J} \subseteq \mathcal{F}(M)$. Then*

$$\mathcal{A} \text{ is stuck in } \mathcal{F}(M) \quad\Longrightarrow\quad \{\mathrm{cl}_N(A) \mid A \in \mathcal{A}\} \text{ is stuck in } \mathcal{F}(N).$$

Proof. Let $k = |\mathcal{A}|$, then we may order the elements of $\mathcal{A} = \{A_1, \dots, A_k\}$ such that $\mathcal{A}_i = \left\{A_j \mid 1 \leq j \leq i\right\}$ is stuck in $\mathcal{F}(M)$ for $i \in \{1, \dots, k\}$, and such that $\cap\mathcal{A}_{i-1}$ either covers or equals $\cap\mathcal{A}_i$ in $\mathcal{F}(M)$ for $i \in \{2, \dots, k\}$. This is equivalent to stating that

$$\mathrm{rk}_M(\cap\mathcal{A}_i) \in \{\mathrm{rk}_M(\cap\mathcal{A}_{i+1}),\, \mathrm{rk}_M(\cap\mathcal{A}_{i+1}) + 1\}$$

holds. If $\mathrm{rk}_M(\cap\mathcal{A}_i) = \mathrm{rk}_M(\cap\mathcal{A}_{i+1})$ holds, then $\cap\mathcal{A}_i = \cap\mathcal{A}_{i+1}$ follows. In this case, $\cap\{\mathrm{cl}_N(A) \mid A \in \mathcal{A}_i\} = \cap\{\mathrm{cl}_N(A) \mid A \in \mathcal{A}_{i+1}\}$ holds and in turn we get

$$\mathrm{rk}_N(\cap\{\mathrm{cl}_N(A) \mid A \in \mathcal{A}_i\}) = \mathrm{rk}_N(\cap\{\mathrm{cl}_N(A) \mid A \in \mathcal{A}_{i+1}\}).$$

If $\mathrm{rk}_M(\cap\mathcal{A}_i) = \mathrm{rk}_M(\cap\mathcal{A}_{i+1}) + 1$ holds, then $\cap\mathcal{A}_i$ covers $\cap\mathcal{A}_{i+1}$. Since both $\cap\mathcal{A}_i$ and $\cap\mathcal{A}_{i+1}$ are stuck in $\mathcal{F}(M)$, we get

$$\begin{aligned} \mathrm{rk}_N(\cap\{\mathrm{cl}_N(A) \mid A \in \mathcal{A}_i\}) &= \mathrm{rk}_M(\cap\mathcal{A}_i) \\ &= \mathrm{rk}_M(\cap\mathcal{A}_{i+1}) + 1 \\ &= \mathrm{rk}_N(\cap\{\mathrm{cl}_N(A) \mid A \in \mathcal{A}_{i+1}\}) + 1. \end{aligned}$$

Therefore $\cap\{\mathrm{cl}_N(A) \mid A \in \mathcal{A}_i\}$ covers $\cap\{\mathrm{cl}_N(A) \mid A \in \mathcal{A}_{i+1}\}$ in $\mathcal{F}(N)$. Thus

$$\mathrm{rk}_N(\cap\{\mathrm{cl}_N(A) \mid A \in \mathcal{A}_i\}) \in \{\mathrm{rk}_N(\cap\{\mathrm{cl}_N(A) \mid A \in \mathcal{A}_i\}) + d \mid d \in \{0, 1\}\}$$

holds, therefore $\{\mathrm{cl}_N(A) \mid A \in \mathcal{A}\}$ is stuck in $\mathcal{F}(N)$. □

4. Persistent Violations

J. Bonin recently showed on *The Matroid Union*'s blog [1], that $P_8^=$ is not a gammoid. The line of argument is the following: $P_8^=$ is obviously not a strict gammoid. Therefore, if $P_8^=$ would be a gammoid, then there would be some finite extension N of $P_8^=$ that is a strict gammoid. By carefully examining the symmetries of $P_8^=$, J. Bonin showed that no matter how $P_8^=$ is extended to N in a rank-preserving way, there is a violation $\mathcal{A} \subseteq \mathcal{Z}(N)$ which proves that N is not a strict gammoid. As a consequence, $P_8^=$ is not a gammoid.

In this section, we introduce an additional property, such that if a violation $\mathcal{A} \subseteq \mathcal{Z}(M)$ has this property, then a copy of it will persevere in any extension of M. Furthermore, this property only depends on M and therefore yields an easy to check sufficient condition that proves M to be outside the class of gammoids.

Definition 15. *Let M be a matroid, $\mathcal{A} \subseteq \mathcal{Z}(M)$ a family of cyclic flats. We define the set of* **positive subfamilies** *of $\mathcal{A}$ to be those subfamilies of $\mathcal{A}$ with odd cardinality. We write*

$$\mathcal{A}^+ = \{\mathcal{F} \subseteq \mathcal{A} \mid |\mathcal{F}| \in 2\mathbb{N} + 1\}.$$

Definition 16. *Let M be a matroid, $\mathcal{A} \subseteq \mathcal{Z}(M)$ a violation. We call $\mathcal{A}$* **persistent**, *if every positive subfamily $\mathcal{F} \in \mathcal{A}^+$ is stuck in $\mathcal{F}(M)$.*

Lemma 17. *Let M be a matroid on E, $\mathcal{A} \subseteq \mathcal{Z}(M)$ be a persistent violation with regard to M, $e \notin E$. Let N be a rank-preserving extension of M on the ground set $E \cup \{e\}$. Then*

$$\{\mathrm{cl}_N(A) \mid A \in \mathcal{A}\} \subseteq \mathcal{Z}(N)$$

is a persistent violation with regard to N.

Proof. First, we prove that $\{\mathrm{cl}_N(A) \mid A \in \mathcal{A}\}$ is indeed a violation in N. From Lemma 8 (iii) we obtain the equation

$$s_N(\{\mathrm{cl}_N(A) \mid A \in \mathcal{A}\}) = s_M(\mathcal{A}) + \sum_{\emptyset \neq \mathcal{A}' \subseteq \mathcal{A}} (-1)^{|\mathcal{A}'|+1} \delta_{\mathcal{J}}(\mathcal{A}').$$

Clearly $s_M(\mathcal{A}) < 0$ since $\mathcal{A}$ is a violation, let $S := \sum_{\emptyset \neq \mathcal{A}' \subseteq \mathcal{A}} (-1)^{|\mathcal{A}'|+1} \delta_{\mathcal{J}}(\mathcal{A}')$. We argue that $S \leq 0$ since for every $\emptyset \neq \mathcal{A}' \subseteq \mathcal{A}$, $(-1)^{|\mathcal{A}'|+1} \delta_{\mathcal{J}}(\mathcal{A}') \leq 0$. If $|\mathcal{A}'| \in 2\mathbb{N}$, we see that

$$(-1)^{|\mathcal{A}'|+1} \delta_{\mathcal{J}}(\mathcal{A}') = -\delta_{\mathcal{J}}(\mathcal{A}') \leq 0$$

obviously holds. Now let $|\mathcal{A}'| \in 2\mathbb{N}+1$, thus $\mathcal{A}' \in \mathcal{A}^+$ is a positive subfamily. Using Lemma 8 (ii) we see that for every $\emptyset \neq \mathcal{A}' \subseteq \mathcal{A}$,

$$\delta_{\mathcal{J}}(\mathcal{A}') = \mathrm{rk}_N(\cap\{\mathrm{cl}_N(A) \mid A \in \mathcal{A}'\}) - \mathrm{rk}_M(\cap \mathcal{A}').$$

Since $\mathcal{A}$ is a persistent violation, we know that such $\mathcal{A}'$ is stuck in $\mathcal{F}(M)$. Therefore, $\mathrm{rk}_N(\cap\{\mathrm{cl}_N(A) \mid A \in \mathcal{A}'\}) = \mathrm{rk}_M(\cap \mathcal{A}')$ by Lemma 13, so $\delta_{\mathcal{J}}(\mathcal{A}') = 0$. Therefore $\{\mathrm{cl}_N(A) \mid A \in \mathcal{A}\}$ is a violation.

We can apply Lemma 14 to see that $\{\mathrm{cl}_N(A) \mid A \in \mathcal{A}\}$ is again a stuck violation:

$$\begin{aligned}\{\mathrm{cl}_N(A) \mid A \in \mathcal{A}\}^+ &= \{\{\mathrm{cl}_N(F) \mid F \in \mathcal{F}\} \mid \mathcal{F} \subseteq \mathcal{A}, |\mathcal{F}| \in 2\mathbb{N}+1\} \\ &= \{\{\mathrm{cl}_N(F) \mid F \in \mathcal{F}\} \mid \mathcal{F} \in \mathcal{A}^+\}.\end{aligned}$$

Thus, every positive subfamily of $\{\mathrm{cl}_N(A) \mid A \in \mathcal{A}\}$ is the closure-image of a positive subfamily of $\mathcal{A}$, and therefore is stuck. □

Corollary 18. *If a matroid M has a persistent violation, then M is not a gammoid.*

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