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Game-Perfect Directed Forests

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Game-Perfect Directed Forests

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Abstract

We consider digraph colouring games where two players, Alice and Bob, alternately colour vertices of a given digraph D with a colour from a given colour set in a feasible way. The game ends when such move is not possible any more. Alice wins if every vertex is coloured at the end, otherwise Bob wins. The smallest size of a colour set such that Alice has a winning strategy is the game chromatic number of D. The digraph D is game-perfect if, for every induced subdigraph H of D, the game chromatic number of H equals the size of the largest symmetric clique of H. In the strong game introduced by Andres [2], colouring a vertex is feasible if its colour is different from the colours of its in-neighbours. In the weak game introduced by Yang and Zhu [22], colouring a vertex is feasible unless it creates a monochromatic directed cycle. There are six variants for each game, which specify the player who begins and whether skipping is allowed for some player. For all six variants of both games, we characterise the class of game-perfect semiorientations of forests by a set of forbidden induced subdigraphs and by an explicit structural description.

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1 Introduction

In this paper, we consider the *strong digraph colouring game* and the *weak digraph colouring game* introduced by Andres [2] and Yang and Zhu [22], respectively.

The strong digraph colouring game was studied in some recent papers [2, 3, 6, 14, 22]. In this game, two players, Alice and Bob, alternately choose a colour c from a given set and colour an uncoloured vertex v of an initially uncoloured, simple and finite digraph D, under the constraint that v does not have any in-neighbour which has been coloured with c. Alice wins if all vertices of D can be coloured finally; otherwise, Bob wins.

Andres [6] considered six variants of the game. Depending on which one we play with, Alice or Bob is the first player, and one of them may be allowed to skip turns. We denote these variants by $g = [X, Y]$. The player $X \in \{A, B\}$ takes the first move and $Y \in \{A, B, -\}$ has the right to skip any number of turns. $A, B, -$ denote Alice, Bob, and none of the players, respectively. The *g-game chromatic number* $\chi_g(D)$ of a digraph D is the smallest natural number t such that Alice has a winning strategy for the strong digraph colouring game with t colours under the g variant.

The concept of game-perfect digraphs was introduced and first studied by Andres [6]. A *symmetric clique* is a digraph such that between any two different vertices u, v the arcs (u, v) and (v, u) exist. The clique number $\omega(D)$ of a digraph D is the number of vertices of the largest symmetric clique in D. It is clear that $\omega(D) \leq \chi_g(D)$ for any D and g, since all vertices of a symmetric clique should have different colours. For each variant g, a digraph D is *g-perfect* if any induced subdigraph H of D has $\omega(H) = \chi_g(H)$.

A non-game analogue of the game chromatic number is the *dichromatic number* of a digraph introduced by Neumann-Lara [20], which is the smallest number of colours used in a (not necessarily proper) colouring of the vertices of the digraph such that the colour classes do not contain monochromatic directed cycles. A digraph D is *perfect* if, for any induced subdigraph H of D, the dichromatic number and clique number of H are equal. Since the dichromatic number is an obvious lower bound on the game chromatic number, any game-perfect digraph is also a perfect digraph. Using the Strong Perfect Graph Theorem [15], which concerns a characterisation of perfect undirected graphs, Andres and Hochstättler [8] characterised perfect digraphs by a set of forbidden induced subdigraphs, which generalizes the Strong Perfect Graph Theorem. In this paper, we consider similar characterisations with respect to games.

By considering undirected graphs as symmetric digraphs, the dichromatic number generalizes the chromatic number of an undirected graph. In the same way, the digraph colouring game is a generalization of the well-known graph colouring game [17] that was made popular by the works of Bodlaender [12] and Faigle et al. [16]. Bounding the game chromatic number for some classes of undirected graphs has provoked many studies; a survey on the first achievements for planar graphs was given by Bartnicki et al. [11]. The best known upper bound on the game chromatic number of planar graphs is 17 and was obtained by Zhu [24]. More references on the general topic of graph colouring games can be found in the recent survey by Tuza and Zhu [21].

Game-perfect undirected graphs were introduced by Andres [4]. A characterisation of game-perfect undirected graphs by forbidden induced subgraphs and by explicit structural descriptions was given by Andres [5] for the games $[B, B]$, $[A, B]$, and $[A, -]$ and recently by Lock [19] and Andres and Lock [9] for the game $[B, -]$.

To deal with digraphs, a *semiorientation* of a graph G is a digraph D on the same vertex set such that every edge vw of G is replaced by either an arc (v, w) or (w, v) or both. Andres [6] proposed the problem of characterising g-perfect digraphs D and partially solved it with respect to the clique number of D. The problem with respect to clique number 1 is trivial and that with respect to clique number 2 was partially solved. For the latter, Andres characterised g-perfect semiorientations of paths, cycles and complete graphs with clique number 2 for all the six variants.

In this paper, we give further results on the characterisation problem with respect to clique number 2. We characterise game-perfect semiorientations of forests, which have clique number at most 2, by a set of forbidden induced subdigraphs and by an explicit structural description. Since paths are forests, our results include the result on semiorientations of paths given by Andres [6]. The two main results of this paper are stated as follows:

Theorem 1. *For a semiorientation D of a forest, the following are equivalent.*

(i) D is $[A, A]$-perfect.

(ii) D does not contain any of the following 24 forbidden configurations (depicted in Figure 5) as an induced subgraph: the 6 in-chairs, the 6 in-brooms, the 2 in-P_5s, F_4, $F_{3,1}$, $F_{3,2}$, F_8, $F_{\rightarrow}^{(1)}$, $F_{\rightarrow}^{(2)}$, $F_{+}^{(1)}$, $F_{+}^{(2)}$, $F_{+}^{(3)}$, $F_{+}^{(4)}$.

(iii) D is either empty or D has a component of one of the types $E_1, \ldots, E_{12}$ (depicted in Figure 6) and every other component of D is a P_4 or a star.

Theorem 2. *For a semiorientation D of a forest, the following are equivalent.*

(i) D is $[A, B]$-perfect.

(i') D is $[A, -]$-perfect.

(ii) D does not contain any of the following 7 configurations as an induced subdigraph: the 3 in-P_4s (see Figure 9), F_4, $F_{3,1}$, $F_{3,2}$, $F_{+}^{(3)}$.

(iii) D is either empty or D has a component of one of the types $E_1^A, \ldots, E_4^A$ (depicted in Figure 10) and every other component of D is a star.

For the games that Bob begins, i.e., $[B, A]$, $[B, -]$, and $[B, B]$, which are much easier to handle, we give similar characterisations in Theorems 41, 42, and 43, respectively, in Section 5. Thus, we characterise game-perfect directed forests for all six possible variants of the strong digraph colouring game.

Yang and Zhu [22] proposed a different digraph colouring game: the weak digraph colouring game. Both games are identical to the undirected graph colouring game when restricted to undirected graphs. A notion of game-perfectness can be defined also for the weak digraph colouring games (cf. [7]). In Section 6, we give characterisations for weakly game-perfect directed forests for any variant of the weak digraph colouring game.

Here is an outline of the rest of this paper. Terminologies and notations will be introduced in Section 2. The proofs of Theorem 1 and 2 will be given in Section 3 and 4, respectively. Section 5 is for the three variants of the strong digraph colouring game that Bob begins. We deal with the weak digraph colouring game for all the six variants in Section 6. In Section 7, we discuss the strong and weak digraph colouring games on directed infinite forests, by extending our results developed for finite forests in the previous sections. Some open questions on digraph colouring games will be discussed in Section 8.

2 Preliminaries

2.1 Basic Notation and Terminology of Digraphs

We consider digraphs of the form (V, A), where V is a finite set of *vertices* and $A \subseteq V \times V \setminus \{(v, v) \mid v \in V\}$ is the set of *arcs*. In particular, this means the digraphs we consider have neither loops nor multiple arcs.

An *in-arc* of a vertex v is (u, v) for some vertex u, an *out-arc* of vertex v is (v, w) for some vertex w. A *single arc* is an arc (u, v) such that (v, u) does not exist. If both (u, v) and (v, u) exist, $uv = \{(u, v), (v, u)\}$ is called an *edge*, and u is called a *symmetric neighbour* of v. An arc is either a single arc or an element of an edge.

A *directed component* of a digraph is a component containing at least one single arc. A *symmetric digraph* is a digraph without single arcs. Therefore, a symmetric digraph can be interpreted as and also called an undirected graph by interpreting the two arcs of every edge as an edge in the context of undirected graphs. The *underlying graph* $G(D)$ of a digraph D is the undirected graph obtained by replacing all single arcs (v, u) in D by the edge vu, which makes D symmetric. A *semiorientation* D of an undirected graph G is any digraph D such that $G(D) = G$. There is no common terminology for the concept of semiorientation. For example, semiorientations are also called *biorientations* (by Bang-Jensen and Gutin [10]) and *superorientations* (by Boros and Gurvich [13]).

For short, in the rest of this paper, we will call any semiorientation of a tree, a forest, or a path simply a *tree*, a *forest*, or a *path*, respectively. Also, a connected induced subdigraph of a semiorientation of a tree will be simply called a *subtree*.

The *symmetric part* $S(D)$ of a digraph $D = (V, A)$ is the digraph (V, E), where E is the union of all edges. In other words, $S(D)$ is the maximal symmetric subdigraph of D.

If (u, v) exists, regardless of the existence of (v, u), u is called an *in-neighbour* of v and v is called an *out-neighbour* of u. The *degree* of a vertex v in a digraph D is the degree of v in $G(D)$.

The *distance* between two arcs (u, v) and (u', v') in a directed tree D, where $\{u, v\} \neq \{u', v'\}$, is denoted by $\mathrm{dist}((u, v), (u', v'))$ and defined as follows. In $G(D)$, which is an undirected tree, there is a unique path with its starting and ending edges being uv and $u'v'$. This path P has length 2 if uv and $u'v'$ are adjacent; otherwise, it has length at least 3. The distance between (u, v) and (u', v') in D is then defined as $\ell - 2$ where ℓ is the length of P. For example, the distance between (a, b) and (c, d) in the subfigure E_3 in Figure 6 is 1; while the distance between (y_1, v) and (y_k, v) in the subfigure E_1 in Figure 6 is 0.

Recall that a *symmetric clique* is a symmetric complete digraph. The *clique number* $\omega(D)$ of a digraph D is the number of vertices of the largest symmetric clique in D.

We refer to the monography of Bang-Jensen and Gutin [10] for undefined terms or notation in this paper. For example, $d^+(v)$ and $d^-(v)$ denote the out-degree and the in-degree of a vertex v, respectively.

2.2 Terminology of Strong Digraph Colouring Games

The following definitions in this Section 2.2 and the definitions in Section 2.3 refer to strong digraph colouring games. Definitions for weak digraph colouring games will be given in Section 6 when they are needed. A *partial colouring* of a digraph is an assignment of colours to some of the vertices. For a strong digraph colouring game g, an uncoloured or a partially coloured digraph D is *k-g-permitted* if Alice has a winning strategy for g played with k colours on D and *k-g-unpermitted* otherwise. An uncoloured digraph D is *g-nice* if $\omega(D) = \chi_g(D)$. Thus, D is g-perfect if and only if every of its induced subdigraphs is g-nice. By definition, a g-nice digraph with clique number k is k-g-permitted.

During a game, colouring a vertex v with colour c is a *Bob-winning move* if v is uncoloured, c is available for v, and colouring v with c makes some out-neighbour of v uncolourable. Two or more Bob-winning moves that exist at the same turn are called *independent* if colouring any single vertex at this turn can eliminate at most one of them.

Therefore, if it is Alice's turn and colouring v with c is a Bob-winning move, Bob can win on his next turn unless Alice colours v or some of its neighbours at this turn. If there exist at least two independent Bob-winning moves, Bob wins the game. These observations will be employed in the proof of Theorem 1.

For any digraph D, the six different games are related in the following way (see [6]).

$$\omega(D) \le \chi(D) \le \chi_{[A,A]}(D) \left\{ \begin{array}{l} \le \chi_{[A,-]}(D) \le \chi_{[A,B]}(D) \le \\ \le \chi_{[B,A]}(D) \le \chi_{[B,-]}(D) \le \end{array} \right\} \chi_{[B,B]}(D). \quad (1)$$

Consequently, if we denote the set of g-perfect digraphs by $\mathcal{GP}_g$ for each g, and the set of perfect digraphs by $\mathcal{P}$, we have

$$\mathcal{GP}_{[B,B]} \left\{ \begin{array}{c} \subseteq \mathcal{GP}_{[A,B]} \subseteq \mathcal{GP}_{[A,-]} \subseteq \\ \subseteq \mathcal{GP}_{[B,-]} \subseteq \mathcal{GP}_{[B,A]} \subseteq \end{array} \right\} \mathcal{GP}_{[A,A]} \subseteq \mathcal{P}. \tag{2}$$

2.3 Threatening Out-Degree

In this subsection, we will introduce the concept of threatening out-degree. The motivation of it is to simplify the proof of Theorem 1 (i)$\Longrightarrow$(ii).

In a digraph D, a vertex v is *safe* if $d^-(v) < \omega(D)$, otherwise it is *unsafe*. We remark that an unsafe vertex might become uncolourable during the game, whereas a safe vertex can always be coloured. In an uncoloured digraph D, the *threatening out-degree* of a vertex v, denoted by $d^+_{\text{thr}}(v)$, is the number of unsafe out-neighbours of v.

For example, in the subfigure with caption $F_{7,1}$ in Figure 4, vertices are attached with their corresponding threatening out-degrees. Safe and unsafe vertices are represented by unfilled and filled vertices, respectively. In $F_{7,1}$, v is safe since v has exactly one in-neighbour and $\omega(F_{7,1}) = 2$. Also, $d^+_{\text{thr}}(v) = 2$ since v has exactly two unsafe out-neighbours. Since the leaf adjacent to u is safe, u has two neighbours but only one unsafe out-neighbour, which implies $d^+_{\text{thr}}(u) = 1$.

Intuitively, the threatening out-degree of a vertex v measures the threat of colouring v to Alice at the beginning of the game. Therefore, at the beginning of any variant where Alice takes the first move, Alice may prefer to skip or colour vertices with threatening out-degree 0. This intuition will be rigorously presented in the following lemma:

Lemma 3. *Let $Y \in \{A, B, -\}$. Let D be a digraph with $\omega(D) \le 2$. For the $[A, Y]$-game played with $\omega(D)$ colours on D, if Alice colours a vertex v with $d^+_{\text{thr}}(v) \ge 1$ in her first move of the digraph colouring game, Bob will win the game. Equivalently, in any of Alice's winning strategies for this game, Alice's first move is either colouring a vertex v with $d^+_{\text{thr}}(v) = 0$ or skipping.*

Proof. If Alice colours a vertex v with $d^+_{\text{thr}}(v) \ge 1$ in her first move, an unsafe out-neighbour of v, denoted by u, will have no available colours in the game played with one colour. For the game with two colours, let again u be an unsafe out-neighbour of v, the vertex Alice has coloured. The vertex u exists, since $d^+_{\text{thr}}(v) \ge 1$. Since u is unsafe and $\omega(D) = 2$, the vertex u has at least two in-neighbours. Therefore, Bob can colour an in-neighbour w of u with $w \ne v$ with the other colour so that u has no available colours. □

The above lemma will be employed in the proof of Theorem 1 (i)$\Longrightarrow$(ii).

2.4 Notation Concerning Structures

By P_n, C_n, K_n, and $(n-1)$-*star* we denote the undirected path, cycle, complete graph and star of n vertices, respectively. The smallest star is the 0-star, which consists of one vertex. An *out-leaf arc* is a single arc (u, v), so that u is the unique neighbour of v. A *k-in-star* is a digraph consisting of $k+1$ vertices and k single arcs which point towards a unique central vertex.

For a vertex v and an integer $k \geq 2$:

- A *pending star at v* is an undirected k-star such that v is a leaf of the star.
- A *P_k at v* is an undirected P_k such that v is a leaf of the P_k.
- A *broken P_k at v* is an undirected P_k such that v is an internal vertex of the P_k.

Moreover, for a vertex v and an integer $k \geq 0$, as depicted in Figure 1:

- A *star* (*k-star*) *at v* is an undirected star (undirected k-star) such that v is the center of the star. (If $k \in \{0, 1\}$, an arbitrary vertex of the star can be considered as center.)
- A *2-gadget at v* is a star or P_3 at v.
- A *P-gadget at v* is a star or pending star at v.
- A *4-gadget at v* is a star, pending star, P_4 or broken P_4 at v.
- A *3-gadget at v* is a 3-star, P_4 or broken P_4 at v.

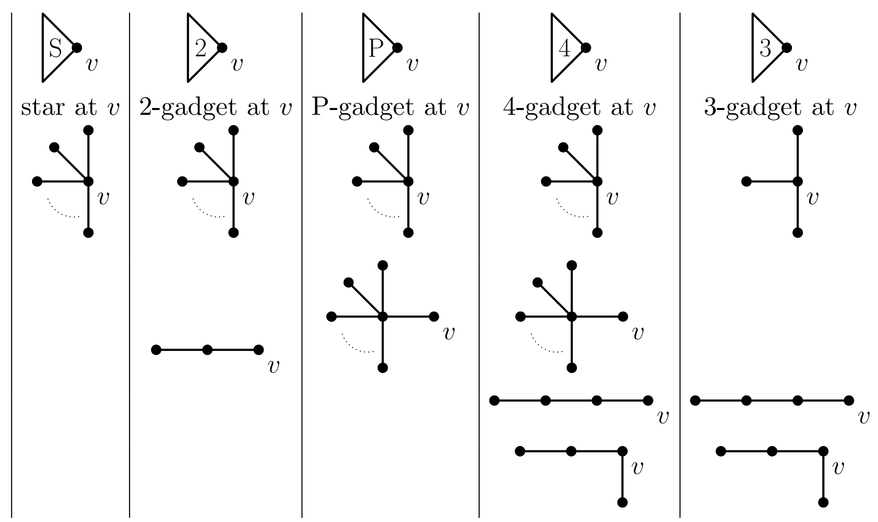

Figure 1: The gadgets and the types of graphs they represent

Let T be a tree and v be one of its vertices. A *v-branch* of T is the subtree induced by v and all vertices of a component of $T - v$. For the v-branch H of T containing a vertex $w \neq v$, we define the *truncated v-branch* containing w, denoted by H_w, as $H_w := H - v$.

2.5 Explanation of the Figures

In the figures of this paper, single arcs are depicted by arrows and edges are depicted by lines. Configurations that might be repeated an arbitrary number of times are indicated by multiple dots. Stars, 2-gadgets, P-gadgets, 4-gadgets and 3-gadgets at a vertex v are depicted by the triangles given in Figure 1.

3 $[A, A]$-Perfect Forests: Proof of Theorem 1

Our method to prove Theorem 1 is inspired by the methods developed in [5], which were also used by Lock [19] and Andres and Lock [9]. We start with an outline of the proof.

Outline of the proof of Theorem 1. In Section 3.1 we will define 24 digraphs, which we call *forbidden digraphs*. In Section 3.2 we will prove that Bob has a winning strategy for the game $[A, A]$ on each of the 24 forbidden digraphs when the number of colours equals its clique number. This means that the forbidden digraphs are not $[A, A]$-perfect, thus (i)$\Longrightarrow$(ii) is proved by contraposition. We will define the 12 classes $E_1, \ldots, E_{12}$ of digraphs, which we call *permitted types*, in Section 3.3. Section 3.4 contains a structural characterisation of forests that do not contain any of the forbidden digraphs as an induced subdigraph. We will first remark that any component of such a forest is a P_4 or a star, except for at most one special component. Then, by a number of case distinctions we will show that, if such a special component exists, the special component must be of one of the permitted types, which proves the implication (ii)$\Longrightarrow$(iii). Finally, in Section 3.5 we will prove that, for any permitted type, every digraph D belonging to this type is $[A, A]$-nice, by describing an explicit winning strategy of Alice, and any subdigraph of D belongs to some permitted type, which together imply that the digraphs of each permitted type are $[A, A]$-perfect, establishing the implication (iii)$\Longrightarrow$(i).

3.1 Forbidden Configurations

In Theorem 1, $[A, A]$-perfect forests are characterised by the thirteen forbidden types of induced subdigraphs shown in Figure 5. These thirteen types totally consist of twenty-four forbidden configurations. All the configurations of the types in-P_5, in-chair and in-broom are depicted in Figures 2, 3 and 4, respectively.

$F_{5,1}$ $F_{5,2}$

Figure 2: The two in-P_5s

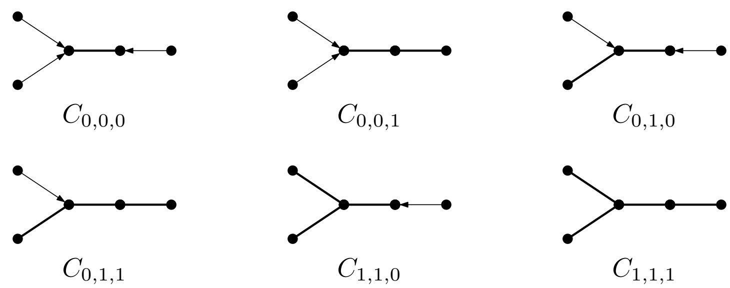

Figure 3: The six in-chairs

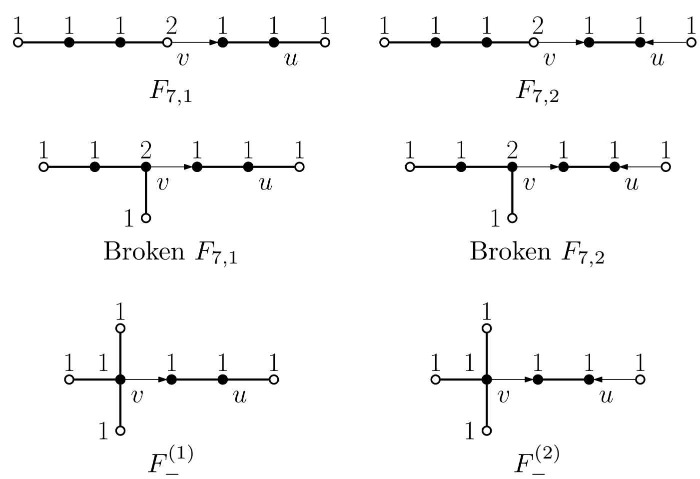

Figure 4: The six in-brooms. In the depictions, unfilled circles denote safe vertices, filled circles unsafe vertices, and the numbers are the threatening out-degree of each vertex.

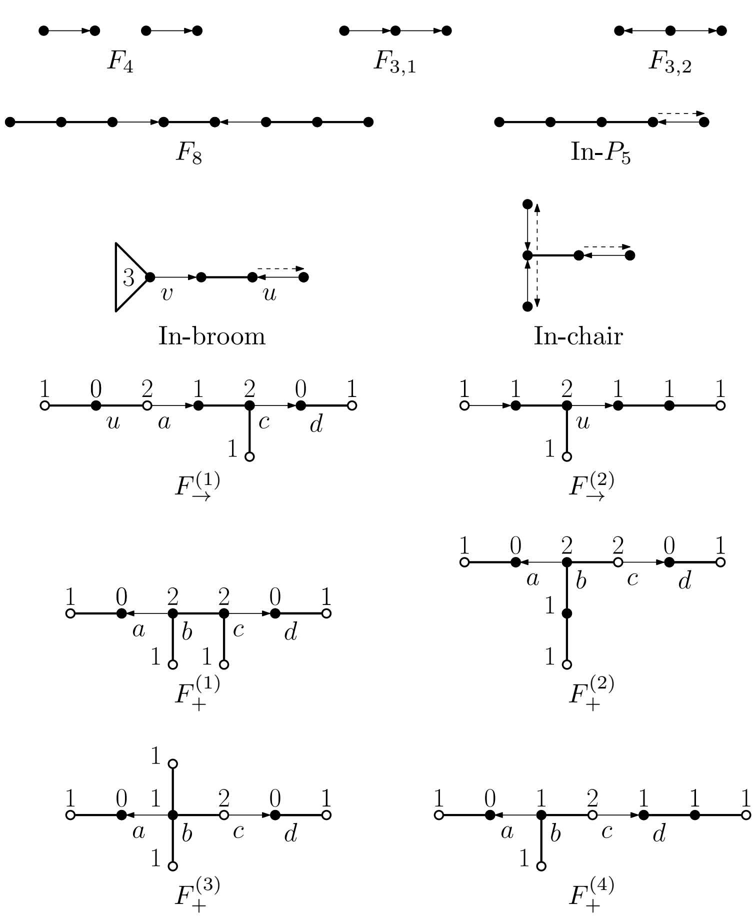

Figure 5: The thirteen types containing twenty-four forbidden configurations for $[A, A]$-perfect digraphs. In the last six depictions, unfilled circles denote safe vertices, filled circles unsafe vertices, and the numbers are the threatening out-degree of each vertex.

3.2 Proof of Theorem 1 (i)$\Longrightarrow$(ii)

It is sufficient to show that every digraph F in the list of forbidden types depicted in Figure 5 is $[A, A]$-*forbidden*, i.e., Bob has a winning strategy for the $[A, A]$-colouring game played on F with $\omega(F)$ colours. We will show them one by one. The case of forests of paths has been already discussed in [6]:

Proposition 4 ([6])**.** *The F_4 and the paths $F_{3,2}$, $F_{3,2}$, F_8 and the two in-P_5s are $[A, A]$-forbidden.*

Note that all in-chairs, all in-brooms, $F_{\to}^{(i)}$ $(1 \leq i \leq 2)$ and $F_{+}^{(i)}$ $(1 \leq i \leq 4)$ have clique number 2. Therefore, in the proofs of the following propositions, we will describe winning strategies of Bob for the $[A, A]$-colouring game with two colours played on these digraphs.

Proposition 5. *The six in-chairs are $[A, A]$-forbidden.*

Proof. For all the six in-chairs, let c be the vertex with degree 3, a and b be its leaf neighbours, d be the other neighbour, and e be the remaining leaf.

If Alice misses her turn, Bob colours a with colour 1. We consider Alice's next move and discuss it by two cases. If Alice colours e or misses her turn, Bob colours b with colour 2. Otherwise, if Alice colours some other vertex x, Bob can colour a vertex y having distance $\text{dist}(x, y) = 2$ with the other colour.

If Alice does not skip in her first move but colours a vertex v, then Bob colours a vertex x having distance $\text{dist}(v, x) = 2$ with the other colour.

In every case, Bob has created a situation with an uncoloured vertex surrounded by two in-neighbours in different colours. Thus, he wins. □

Proposition 6. *The six in-brooms are $[A, A]$-forbidden.*

Proof. Note that every vertex in any in-broom has threatening out-degree at least 1 (see Figure 4). Therefore, by Lemma 3, Alice's first move in any of her winning strategies on an in-broom is skipping.

Then, if the in-broom contains a P_4 or a broken P_4 at v, Bob can win by colouring v in his first move to generate two independent Bob-winning moves.

Otherwise, when the in-broom contains a 3-star at v, Bob may colour a leaf adjacent to v with colour 1. Since colouring anyone of the remaining two leaves adjacent to v with colour 2 is a Bob-winning move, Alice must colour v with colour 2 in her second turn. Then, Bob will win after he colours u with 1. Therefore, Alice has no winning strategy. □

Proposition 7. *$F_{\to}^{(1)}$ and $F_{\to}^{(2)}$ are $[A, A]$-forbidden.*

Proof. Suppose Alice has a winning strategy for $F_{\to}^{(1)}$. Since only u and d have threatening out-degree 0, in her first move, by Lemma 3, she colours u, d or skips. If she colours u, Bob may colour c with the same colour to generate two independent Bob-winning moves. For the remaining two choices of her first move, Bob may colour a to generate two independent Bob-winning moves.

Consider $F_{\rightarrow}^{(2)}$. Since all the vertices have non-zero threatening out-degree, by Lemma 3, Alice's first move in any of her winning strategies is skipping. Then, Bob can win by colouring u in his first move to generate two independent Bob-winning moves. □

Proposition 8. *$F_{+}^{(1)}$, $F_{+}^{(2)}$, $F_{+}^{(3)}$, $F_{+}^{(4)}$ are $[A, A]$-forbidden.*

Proof. We have $d_{\text{thr}}^{+}(a) = 0$ for all the 4 graphs and $d_{\text{thr}}^{+}(d) = 0$ for $F_{+}^{(1)}$, $F_{+}^{(2)}$, $F_{+}^{(3)}$ and all other vertices have nonzero threatening out-degree. Therefore, in Alice's first move of any of her winning strategies for $F_{+}^{(1)}$, $F_{+}^{(2)}$, $F_{+}^{(3)}$, she colours a, d or skips. Her first move of her winning strategy for $F_{+}^{(4)}$ is either colouring a or skipping.

For all the 4 digraphs, if Alice colours a or skips in her first move, Bob may colour c to generate two independent Bob-winning moves.

If Alice colours d in her first move on the digraphs $F_{+}^{(1)}$ and $F_{+}^{(2)}$, Bob may colour b to generate two independent Bob-winning moves.

If Alice colours d in her first move on $F_{+}^{(3)}$, Bob colours c. To avoid the two threats of b given by the leaf neighbours of b, Alice must colour b. Then Bob colours the leaf adjacent with a with the other colour and wins. □

3.3 Permitted Structures

The main permitted type of digraphs for $[A, A]$-perfect digraphs is type E_1. We say a digraph is of type E_1 if it is a connected induced subdigraph of a *multiple in-star*, which is a digraph built from an edge (v, x) by adding a (non-symmetric) out-neighbour z to x, and by possibly adding a 2-gadget at z, some leaf edges incident to v and some (non-symmetric) in-neighbours $y_1, \ldots, y_k$ to v with a 4-gadget at each of them. Note that, by definition, in a digraph of type E_1 the vertices v, x or z need not exist.

The other types $E_2, \ldots, E_{12}$ are more special types not fitting to the definition of a multiple in-star. The permitted types of $[A, A]$-perfect digraphs are depicted in Figure 6. In this figure, unfilled circles indicate optional vertices, whereas filled circles indicate the vertices compulsory for a digraph to be of the type considered.

3.4 Proof of Theorem 1 (ii)$\Longrightarrow$(iii)

3.4.1 Preliminary lemmas

Lemma 9. *If, in a tree that contains no induced F_4, there is a single arc $\vec{e}_1$ and an arc $\vec{e}_2$ with $dist(\vec{e}_1, \vec{e}_2) \geq 2$, then $\vec{e}_2$ is part of an edge.* □

Lemma 10. *In any tree that does not contain $F_{3,2}$, every vertex is incident with at most one single out-arc.* □

Lemma 11. *An undirected tree T that does neither contain P_5 nor the chair, is either the P_4 or a star.*

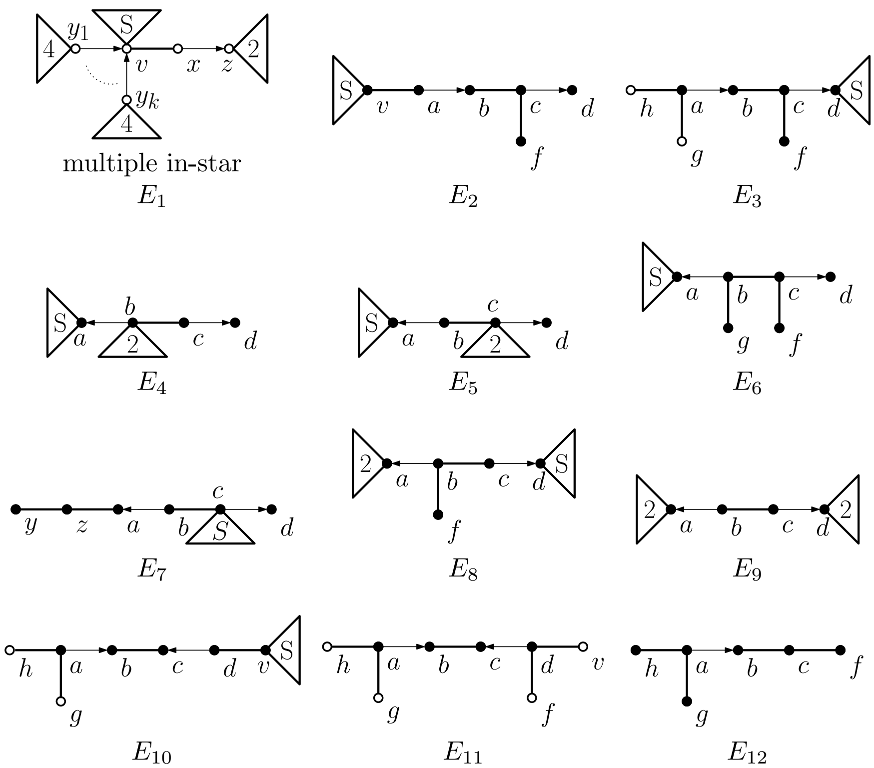

Figure 6: The permitted types. In E_1, every vertex could be optional, under the constraint that E_1 is an induced connected subdigraph of the configuration depicted above.

Proof. Since P_5 is not contained in T, the diameter of T is at most 3. Since no in-chair is contained in T, the tree T is a path when its diameter is 3. Therefore, T is either a P_4 or a star. □

Lemma 12 (Out-Arc-4-Gadget Lemma). *Let (v, w) be a single arc in a tree for which (ii) holds. Assume that the truncated w-branch H_v containing v does not contain a single arc. Then H_v is a 4-gadget at v.*

Proof. Since (ii) is true, H_v does not contain a P_5 nor a chair. By Lemma 11, H_v is a star or P_4, thus H_v is either a star, pending star, P_4 or broken P_4 at v. □

Lemma 13 (In-Arc-2-gadget Lemma). *Let (v, w) be a single arc in a tree for which (ii) holds. Assume that the truncated v-branch H_w containing w does not contain a single arc. Then H_w is a 2-gadget at w.*

Proof. As in the proof of Lemma 12, since (ii) is true and by Lemma 11, H_w is a star, pending star, P_4 or broken P_4 at w. Since the in-P_5 is forbidden, H_w is not a P_4 at w. Since the in-chair is forbidden, H_w is neither a broken P_4 nor a pending k-star at w with $k \geq 3$. If it is a pending 2-star at w, it is a P_3 at w. Thus, H_w is either a star or a P_3 at w. □

3.4.2 Proof of Theorem 1 (ii)⟹(iii): Case analysis

Proof of Theorem 1 (ii)⟹(iii). Let D be a semiorientation of a forest that does not contain any of the 24 forbidden configurations from (ii) as induced subdigraph. Since F_4 is forbidden in D, at most one component of D contains a single arc. By Lemma 11, every other component is a star or a P_4. If D contains no single arc, the proof is complete. Otherwise let T be the component of D containing a single arc.

By a case distinction we prove that T is of one of the types $E_1, \ldots, E_{12}$.

Lemma 14. *Either T contains a vertex with at least two single in-arcs or, T has at most two single arcs and if there are two, they have distance 1.*

Proof. Since F_4 is forbidden in T, any pair of single arcs has distance 1 or is adjacent. Since $F_{3,1}$ and $F_{3,2}$ are forbidden in T, two adjacent arcs are two in-arcs of the same vertex. In a tree it is not possible to have three single arcs which are pairwise at distance 1. □

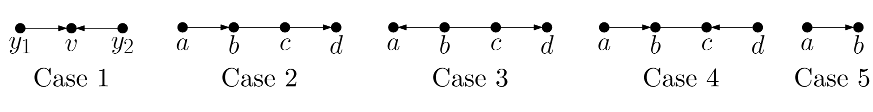

Figure 7: The five cases: T contains a vertex with at least two single in-arcs in Case 1, but not in Cases 2-5. In Cases 2-4, T contains exactly two single arcs; in Case 5, T contains exactly one single arc.

By Lemma 14, we consider the five cases shown in Figure 7.

We now describe our approach for showing that T, in each case, is of one the types E_i. In each case, we first employ the preliminary lemmas to restrict the possible configurations of T; then, we eliminate some of these configurations by using the assumption (ii) that T does not contain any forbidden type. After that, we point out that all remaining configurations have some structure E_i.

Case 1. *The tree T has a vertex v incident with at least two single in-arcs.*

We aim to show that T, in this case, is of type E_1. Assume v has k single in-arcs, say $(y_1, v), \ldots, (y_k, v)$, and $k \geq 2$. For each one of them, say (y, v), the truncated v-branch H_y containing y does not contain any single arc, since otherwise the existence of such a single arc (a, b) would imply that T contains $F_{3,1}$ or $F_{3,2}$ (induced by the vertices a, b, v) or F_4 (induced by the vertices a, b, v, y_i for some i with $y_i \neq y$). Thus, H_y is a 4-gadget by Lemma 12. Observe that:

- v has no out-arc, since otherwise the existence of such an out-arc, say (v, z), would imply that T contains $F_{3,1}$ (induced by the vertices y_1, v, z);
- no symmetric neighbour x of v is incident to an in-arc or another edge than (v, x), since otherwise the existence of such an edge, say xz, or such an in-arc, say (z, x), would imply that T contains an in-chair (induced by the vertices y_1, y_2, v, x, z);
- no symmetric neighbour x of v is incident to more than one out-arc, since otherwise, by Lemma 10, T would contain an induced $F_{3,2}$;
- there is at most one symmetric neighbour of v incident to an out-arc, since otherwise, by Lemma 9, T would contain an induced F_4.

If v has a symmetric neighbour, say x, which has an out-neighbour, say z, then the truncated x-branch H_z containing z does not contain a single arc, since otherwise the existence of such a single arc, say (a, b), would imply T contains F_4 (induced by the vertices a, b, v, y_1). Thus, by Lemma 13, H_z is a 2-gadget. Therefore, in Case 1, T is of type E_1.

In the following Cases 2-5 we explicitly exclude Case 1, i.e. we assume that there is no vertex in T incident with two single in-arcs. By Lemma 14, then T contains at most two single arcs.

Case 2. *The tree T has single arcs (a, b) and (c, d) and an edge bc.*

We aim to prove that, in this case, T is of type E_1, E_2 or E_3. First observe that, by Lemmas 12 and 13, the truncated b-branch H_a containing a is a 4-gadget and the truncated c-branch H_d containing d is a 2-gadget. Moreover, consider T', the component containing a of $T - (c, d)$. Since (ii) is true for T, it is true for T', and so by Lemma 13, the truncated a-branch H'_b containing b of T' is a 2-gadget, i.e. it is either a star at b or a P_3 at b. Observe that, if H'_b is a star at b, then T is of type E_1. In the following we assume H'_b is a P_3 at b, i.e. c has a symmetric neighbour f other than b.

We make the following observations:

- H_a does not contain a 3-gadget, since otherwise the 3-gadget together with b, c and f would induce an in-broom. So H_a is either a pending star at a or a k-star at a for some $k \leq 2$.

- If H_a is a pending star at a, then d is a leaf, since otherwise the existence of a symmetric neighbour g of d would imply that T contains $F_{\rightarrow}^{(1)}$ (induced by the vertices b, c, d, f, g and a P_3 of H_a containing a). Therefore, T is of type E_2.

- H_d is a star at d, since otherwise it would be a P_3 at d, which would imply that T contains $F_{\rightarrow}^{(2)}$ (induced by H_d and the vertices a, b, c, f).

- If H_a is a k-star at a, then as said $k \leq 2$ and T is of type E_3.

Thus, in Case 2 we only get the structures E_1, E_2 and E_3.

Case 3. *The tree T has single arcs (b, a) and (c, d) and an edge bc.*

We aim to prove that, in this case, T is of type E_4, E_5, E_6, E_7, E_8 or E_9. By Lemma 13, the truncated b-branch H_a containing a and the truncated c-branch H_d containing d are both 2-gadgets. We denote by T' the component of $T - (c, d)$ containing a and by T'' the component of $T - (b, a)$ containing c. By Lemma 12, the truncated a-branch containing b in T' and the truncated d-branch containing c in T'' are both 4-gadgets.

Observe that, by definition of a 4-gadget, this is equivalent to say that in T there are a 2-gadget H_b at b and a 2-gadget H_c at c, and that either one of them is reduced to a single vertex or both are reduced to a single pending edge. If both are reduced to a single vertex, then T is of the permitted type E_9; we assume in the following it is not the case.

Moreover, if H_a is a star at a and H_d reduced to a single vertex, then T is of one of the permitted types E_4, E_5 or E_6. By symmetry, we obtain the same permitted types if H_a is a single vertex and H_d is a star. We are left to consider the cases that neither H_a nor H_d is trivial or one of them is a P_3 at its vertex.

- If neither H_a nor H_d is trivial, then, since $F_{+}^{(1)}$, $F_{+}^{(2)}$ and $F_{+}^{(3)}$ are forbidden, H_b or H_c has to be trivial and the non-trivial one, say H_b, has to be reduced to a single edge. Since $F_{+}^{(4)}$ is forbidden, H_d cannot be a P_3 at d and so T is of the permitted type E_8.

- If otherwise H_a or H_d is a P_3 at its vertex, say H_a is a P_3 at a and H_d is trivial, then, since T has no induced in-broom, we conclude the following:

  - H_b or H_c has to be trivial (otherwise H_c, H_b and H_a would induce a broken $F_{7,1}$).
  - H_b cannot be a P_3 at b (otherwise c, H_b and H_a would induce again a broken $F_{7,1}$).
  - H_c cannot be a P_3 at c (otherwise H_c, b, and H_a would induce $F_{7,1}$).

– If H_b is a k-star at b, then $k \leq 1$ (otherwise the digraph induced by c, H_b and H_a would contain an induced $F_-^{(1)}$).

Thus, by what we already stated about H_b and H_c, either H_b is trivial and H_c is a star or H_b is reduced to an edge and H_c is trivial. In the former, T is of the permitted type E_7. In the latter, T is of the permitted type E_8.

Thus, in Case 3 we only get the structures E_4, E_5, E_6, E_7, E_8 and E_9.

Case 4. *The tree T has single arcs (a, b) and (d, c) and an edge bc.*

We aim to prove that, in this case, T is of type E_{10} or E_{11}. First we remark that b and c do not have neighbours outside the set $\{a, b, c, d\}$: If b has another symmetric neighbour than c, say f, then the vertices a, b, c, d, f induce an in-chair, which contradicts (ii). The same is true if we change the roles of b and c.

By Lemma 12 the truncated b-branch H_a containing a and the truncated c-branch H_d containing d are both 4-gadgets. Neither of them is a P_4 or broken P_4 or k-star at their vertex for some $k \geq 3$, since otherwise it would imply that T contains an in-broom ($F_{7,2}$, a broken $F_{7,2}$ or $F_-^{(2)}$, respectively), contradicting (ii). Thus they are either k-stars, for $k \leq 2$, or pending stars at their vertices.

- If one of them, say H_d, is a pending star, then H_a is not also a pending star, since otherwise T would contain F_8 (induced by the vertices b, c and a P_3 of each of H_a and H_d containing a and d, respectively). Thus T is of the permitted type E_{10}.
- If both of them are k-stars, $k \leq 2$, then T is of the permitted type E_{11}.

Thus, in Case 4 we only get the structures E_{10} and E_{12}.

Case 5. *The tree T has a single arc (a, b).*

We aim to prove that, in this case, T is of type E_1 or E_{12}. By Lemma 12, the truncated b-branch H_a containing a is a 4-gadget, and, by Lemma 13, the truncated a-branch H_b containing b is a 2-gadget. We distinguish two cases:

- If H_b is a star at b, then T is of type E_1.
- If H_b is a P_3 at b, then H_a is neither a P_4 nor a broken P_4 nor a k-star at a for some $k \geq 3$, since otherwise, by a similar argument as in Case 4, it would imply that T contain an in-broom ($F_{7,1}$, broken $F_{7,1}$ or $F_-^{(1)}$, respectively). Thus in this case, either H_a is a 2-star at a and T is of permitted type E_{12}, or H_a is a 1-star or a pending star at a and T is of the permitted type E_1.

Thus, in Case 5 we only get the structures E_1 and E_{12}.

This completes the proof of (ii)$\Longrightarrow$ (iii). □

3.5 Proof of Theorem 1 (iii)$\Longrightarrow$(i)

We start with an outline of the proof.

Proof of Theorem 1(iii)$\Longrightarrow$(i). First, we prove that every digraph for which (iii) is true is $[A,A]$-nice (Lemma 15). Thus we will prove that (iii) is also true for every one of its subdigraphs (Lemma 16). This implies that D is $[A,A]$-perfect. □

Lemma 15. *For any i, the disjoint union of a digraph of type E_i and any number of stars and P_4s is $[A,A]$-nice.*

Lemma 16. *If D is a digraph for which (iii) is true and D' is a non-empty subdigraph of D, then (iii) is true for D'.*

In the following we will prove Lemma 15 and Lemma 16. For the proof of Lemma 15, we begin with a folklore observation.

Observation 17. *Every star, the P_4, every star with its center having an additional in-arc and the P_4^- (depicted in Figure 9) are $[B,A]$-nice and thus $[A,A]$-nice.*

Proof. The 0- and the 1-star are complete graphs and thus $[A,A]$-nice. Therefore, it is sufficient to describe a winning strategy for Alice in the game $[B,A]$ played with two colours for the remaining graphs aforementioned.

For any star and for any star with its center having an additional in-arc, if Bob did not colour the central vertex on his first move, then Alice does it right after. Every other vertex has degree 1 and so is always colourable.

For the P_4 and the P_4^-, say Bob colours some vertex u in his first move. Then Alice will colour the unique vertex v such that $\text{dist}(u,v)=2$ with the same colour. Alice wins because the other two vertices must be coloured finally in the other colour. □

Lemma 18 (Arc Deletion Rule)**.** *Let $X \in \{A,B\}$. For any tree T with clique number 2 and an out-leaf arc (u,v), T is $[X,A]$-nice if and only if $T-v$ is $[X,A]$-nice.*

Proof. ($\Leftarrow$): Suppose Alice has a winning strategy for the $[X,A]$-colouring game on $T-v$. During the game on T, if Bob never colours the leaf v, Alice may use her strategy for $T-v$; she will win because v has only one in-neighbour, so v can always be coloured in the game with two colours. If Bob colours v at some turn, Alice skips her next turn. Because v has no out-neighbours, the colouring of v will not affect the subsequent colouring of any vertices in $T-v$. So, after the skip, Alice can resume her winning strategy for $T-v$.

($\Rightarrow$): Suppose Alice has a winning strategy for the $[X,A]$-colouring game on T. During the game on $T-v$, Alice may use her strategy for T. This strategy fails only if at some Alice's turn, she chooses to colour the leaf v in her strategy for T but now v is deleted. In this case, she may skip her turn and resume the strategy starting from her next turn. Because in the game on T, the colouring of v does not affect any subsequent colouring of the other vertices, Alice can successfully resume her strategy and win the game. □

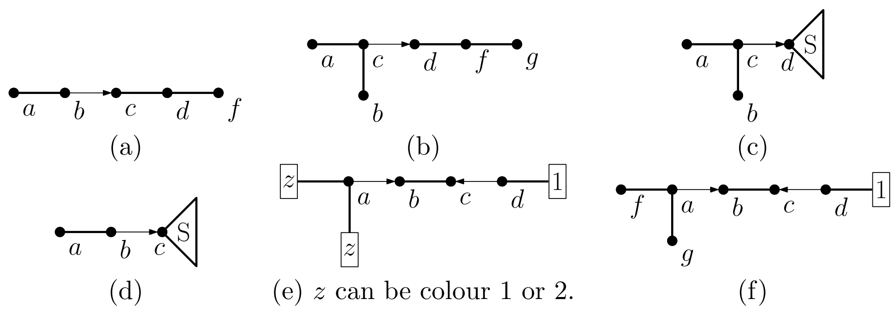

Figure 8: Some 2-$[A, A]$-permitted types. The variable or number inside a rectangle is the colour that has been put on the vertex.

Recall that for game g, an uncoloured or partially coloured digraph G is *k-g-permitted* if Alice can win g with k colours on G, and *k-g-unpermitted* otherwise.

Lemma 19 (P_5-Lemma)**.** *The partially coloured path in Figure 8(a), where the vertex a is coloured, is 2-$[B, A]$-permitted and so 2-$[A, A]$-permitted.*

Proof. We present a winning stategy for Alice in game $[B, A]$. If Bob colours b, then Alice colours d with the same colour. If Bob colours d, then Alice colours c with the other colour. If Bob colours c, then Alice colours f with the same colour, and vice versa. After that Alice wins. □

Lemma 20. *The path in Figure 8(a) is $[B, A]$-nice and so is $[A, A]$-nice.*

Proof. Alice may respond to Bob's first move as follows.

If Bob colours a, then, by the P_5-Lemma (Lemma 19), the resulting partially coloured subtree is 2-$[B, A]$-permitted. If Bob colours b, Alice colours d with the same colour, and vice versa. If Bob colours c, Alice colours f with the same colour, and vice versa. □

Lemma 21. *The digraph in Figure 8(b) is 2-$[B, A]$-permitted and so is [A,A]-nice.*

Proof. We consider all possible first moves of Bob.

If Bob colours a, Alice may colour b with the same colour. After that, Alice may use her winning strategy in the P_5-Lemma (Lemma 19).

If Bob colours c, then Alice may colour f with the same colour, and vice versa, so that the remaining vertices must be coloured finally.

If Bob colours d, Alice colours g with the same colour, and vice versa, so that f must be coloured finally. The generated uncoloured P_3 induced by a, b, c is $[B, A]$-nice. □

Lemma 22. *The digraph in Figure 8(c) is 2-$[B, A]$-permitted and so is $[A, A]$-nice.*

Proof. We consider all possible first moves of Bob. If Bob colours a (resp. b), then Alice colours b (resp. a) with the same colour. After that, d is the unique vertex which may not be coloured finally, and Alice may colour it in her next move if it is not coloured by Bob. If Bob colours c (resp. d), then she may colour d (resp. c) with the other colour, so that all the remaining vertices must be coloured finally. If Bob colours any leaf adjacent to d, then she may colour d to generate a star with the central vertex coloured, and an uncoloured P_3. □

Lemma 23. *(i) The partially coloured subtree in Figure 8(d) with coloured a is 2-$[B, A]$-permitted and so 2-$[A, A]$-permitted.*

(ii) The subtree in Figure 8(d) is $[B, A]$-nice and so is $[A, A]$-nice.

Proof. Since a, b and all the leafs adjacent to c have only one in-neighbour, c is the only vertex which may not be coloured finally. Therefore, Alice may colour c in her first move if Bob does not do so in his first move. □

Lemma 24. *The partially coloured subtree in Figure 8(e) is 2-$[B, A]$-permitted.*

Proof. Alice may respond to Bob's first move as follows.

If Bob colours a, then she may colour c with the same colour, and vice versa, so that the remaining uncoloured vertices b and d must have available colours. If Bob colours b, then she may colour c with the other colour. Thus, the remaining uncoloured vertices a and d must have available colours. If Bob colours d with 2, then she may colour b with 2 so that the remaining uncoloured vertices a and c must have available colours. □

Lemma 25. *The partially coloured subtree in Figure 8(f) is 2-$[B, A]$-permitted.*

Proof. Alice may respond to Bob's first move as follows.

If Bob colours f, then she may colour g with the same colour to generate the 2-$[B, A]$-permitted subtree in Figure 8(e). If Bob colours a, then she may colour c with the same colour, and vice versa, so that the remaining uncoloured vertices b and d must have available colours. If Bob colours b, then she may colour c with the other colour. Thus, the remaining uncoloured vertex d must have an available colour and a, f, g induce a P_3, which is $[B, A]$-nice. If Bob colours d with 2, then she may colour b with 2 so that a similar situation arises as in the previous case. □

Proposition 26. *Every digraph of type E_1 is $[A, A]$-nice.*

Proof. Let H be a digraph of type E_1. If $\omega(H) = 1$, then, since H is connected, H is a r-in-star for some non-negative r. Note that all the r leaves are safe. Therefore, the winning strategy of Alice for the game on this in-star with one colour is to colour the sink in her first move.

Now we may assume $\omega(H) = 2$. In the following the vertex names refer to Figure 6. We first consider the case that the vertex v exists. Then Alice may colour the vertex v in her first move to generate some uncoloured or partially coloured subtrees. Observe that any generated subtree must be a P_4, a star, a star with the central vertex coloured or one of the subtrees in Figure 8(a) or 8(d) with vertex a coloured. By Observation 17, Lemma 19 and Lemma 23, each such subtree is 2-$[B, A]$-permitted. Then, Alice will eventually win the game if she employs the following strategy in the rest of the game. Suppose Bob acted on some subtree T in his last move. If T is not fully coloured yet, she acts on T according to her winning strategy for the game $[B, A]$ on T with two colours; if T is fully coloured, she passes her turn.

Second consider the case that v does not exist. Since H is connected, H is either the P_4, a star, a star whose center has an additional in-arc (when the 2-gadget is a star), or the P_4^- (when the 2-gadget is a P_3) which is depicted in Figure 9. All of them are $[A, A]$-nice by Observation 17. □

Proposition 27. *Every digraph of type E_2 or E_7 is $[A, A]$-nice.*

Proof. With the Arc Deletion Rule (Lemma 18), we may consider the games on E_2 and E_7 with their out-leaf arcs deleted. In her first move, Alice may colour v (for the game on E_2) or c (for the game on E_7) to generate the path in Figure 8(a) with a coloured. This subtree is 2-$[B, A]$-permitted by the P_5-Lemma (Lemma 19). □

Proposition 28. *Every digraph of type E_3 or E_8 is $[A, A]$-nice.*

Proof. In her first move, Alice may colour d to generate a star with the central vertex coloured and either a P_4^- (see Figure 9) or the subtree in Figure 8(a), 8(b) or 8(c). By the same strategy as in the proof of Lemma 19 or by Lemma 20, Lemma 21 or Lemma 22 Alice wins, respectively. □

Proposition 29. *Every digraph of type E_4, E_5 or E_6 is $[A, A]$-nice.*

Proof. With the Arc Deletion Rule (Lemma 18), we may consider the game on a digraph of type E_4, E_5 or E_6 with its out-leaf arc (c, d) deleted, which is a digraph of type E_1. By Proposition 26, Alice wins. □

Proposition 30. *Every digraph of type E_9 is $[A, A]$-nice.*

Proof. Alice may skip her first move and respond to Bob's first move as follows. By the structural symmetry of this digraph, it is sufficient to consider the cases when Bob plays on the left half of the digraph

If Bob colours b, then she may colour a symmetric neighbour of a with the same colour so that when the 2-gadget at a is P_3, they totally generate a partially coloured P_2, a partially coloured star with the central vertex a having an available colour, and the partially coloured subtree in Figure 8(a) or 8(d) with a coloured; when the 2-gadget at a is a star, they totally generate a partially coloured star with the central vertex a having an available colour, and the partially subtree in Figure 8(a) or 8(d) with a coloured.

In the following we consider the case that Bob colours a vertex in the 2-gadget at a.

When the gadget is $P_3 = yza$, if Bob colours y (resp. a), then she may colour a (resp. y) with the same colour to generate a partially coloured P_3 with the central vertex z having an available colour, and the uncoloured subtree in Figure 8(a) or 8(d). If Bob colours z, then she may colour a so that they totally generate a partially coloured P_2 and the uncoloured subtree in Figure 8(a) or 8(d).

When the gadget is a star, if Bob colours a leaf adjacent to a (resp. a), then she may colour a (resp. a leaf adjacent to a) so that they totally generate a partially coloured star with the central vertex a coloured, and the uncoloured subtree in Figure 8(a) or 8(d). □

Proposition 31. *Every digraph of type E_{10} is $[A, A]$-nice.*

Proof. We only discuss the case that both optional vertices g and h exist, the strategies for the other cases are very similar.

Alice may colour v with 1 to generate a star with the central vertex coloured and the partially coloured subtree in Figure 8(f). By Lemma 25, the latter is 2-$[B, A]$-permitted. □

Proposition 32. *Every digraph of type E_{11} is $[A, A]$-nice.*

Proof. Again, we only discuss the case that all four optional vertices exist, the strategies for the other cases being very similar.

Alice may skip her first move and respond to Bob's first move as follows. By the structural symmetry of this digraph, we may consider the cases when Bob plays on the left half of the digraph.

If Bob colours f, then Alice may colour v with the same colour to generate a partially coloured subtree, denoted by T. After that, the subtree induced by all the uncoloured vertices (a, b, c, d, h, g) of T is the same as that induced by all the uncoloured vertices (a, b, c, d, h, g) of the partially coloured subtree in Figure 8(f). Moreover, any two vertices with the same label in the two induced subtrees have the same set of available colours. Therefore, the games on T and the partially coloured subtree in Figure 8(f) are equivalent. Consequently, T is also 2-$[B, A]$-permitted.

If Bob colours d, then Alice may colour b with the same colour so that they totally generate a star with the central vertex d coloured, an uncoloured P_3 and a subtree in which c must have an available colour.

If Bob colours c with 1, then Alice may colour b with 2 so that they totally generate two uncoloured P_3 and a completely coloured P_2. □

Proposition 33. *The digraph of type E_{12} is $[A, A]$-nice.*

Proof. This was proven in Lemma 21. □

Proof of Lemma 15. Alice has the following winning strategy with 2 colours for the game $[A, A]$ played on the disjoint union of a digraph D_0 of type E_i and stars $S_1, \ldots, S_p$ and P_4s $P_1, \ldots, P_q$, where $p, q \geq 0$.

Type E_1	
x	comp. of $H-x$
4	$S/K_1^{\cup}, E_1$
y_i	$S/[K_2, K_1]/K_1^{\cup}, E_1$
v	$S^{\cup}, P_4^{\cup}, K_1^{\cup}, (E_1)$
S	E_1
x	$E_1, (S)$
z	$E_1, K_2/K_1^{\cup}$
2	$E_1, (K_1)$

Type E_2	
x	comp. of $H-x$
S	E_2
v	$K_1^{\cup}, E_3$
a	S, E_1
b	S, E_1
c	E_1, K_1, K_1
d	E_1
f	E_1

Type E_3	
x	comp. of $H-x$
a	$(K_1), (K_1), E_1$
b	S, E_1
c	E_1, K_1, S
d	$E_1/E_{12}, K_1^{\cup}$
S	E_3
f	E_1
g	E_3 by def.
h	E_3 by def.

Type E_4	
x	comp. of $H-x$
S	E_4
a	$(K_1 \cdots K_1), E_1$
b	$S, K_2/K_1^{\cup}, E_1$
c	E_1, K_1
d	E_1
2	$(K_1), E_4$

Type E_5	
x	comp. of $H-x$
S	E_5
a	$(K_1 \cdots K_1), E_1$
b	S, E_1
c	$E_1, K_2/K_1^{\cup}, K_1$
d	E_1
2	$E_5, (K_1)$

Type E_6	
x	comp. of $H-x$
S	E_6
a	$K_1^{\cup}, E_1$
b	S, K_1, E_1
c	E_1, K_1, K_1
d	E_1
f	E_4
g	E_5

Type E_7	
x	comp. of $H-x$
y	E_5
z	K_1, E_5
a	K_2, E_1
b	S, E_1
c	E_1, K_1, K_1
d	E_1
f	E_9

Type E_8	
x	comp. of $H-x$
2	$E_8, (K_1)$
a	$E_1, K_2/K_1^{\cup}$
b	E_1, K_1, S
c	S, E_1
d	$K_1^{\cup}, E_1/E_{12}$
f	E_9
S	E_8

Type E_9	
x	comp. of $H-x$
2	$(K_1), E_9$
a	$K_2/K_1^{\cup}, E_1$
b	S, E_1
(use symmetry)	

Type E_{10}	
x	comp. of $H-x$
a	$(K_1), (K_1), E_1$
b	S, E_1
c	E_1, S
d	E_1, S
v	$E_9, K_1^{\cup}$
S	E_{10}
h	E_{10} by def.
g	E_{10} by def.

Type E_{11}	
x	comp. of $H-x$
a	$(K_1), (K_1), E_1$
b	S, E_1
f	E_{11} by def.
g	E_{11} by def.
(use symmetry)	

Type E_{12}	
x	comp. of $H-x$
a	K_1, K_1, S
b	S, K_2
c	E_1, K_1
f	E_1
g	E_1
h	E_1

Table 1: Listing the types of the components of $H-x$ for any digraph H from each type E_i and any vertex x from H

By Observation 17, she has a winning strategy for the game $[B, A]$ on each of $S_1, \dots, S_p, P_1, \dots, P_q$. By Propositions 26–33, she has a winning strategy for D_0. Alice combines these strategies in the following way.

In her first move she acts according to her winning stategy for D_0 (this act might be a skip if required by her strategy). After that, whenever Bob plays on one of the components $D_0, S_1, \dots, S_p, P_1, \dots, P_q$, Alice acts according to her winning strategy for this component on this component, unless the component is fully coloured. In case such a component is fully coloured Alice misses her turn.

Since the colouring of a component does not affect the colouring of any other component, Alice will win finally. □

Proof of Lemma 16. In Table 1, for each digraph H of type E_i and each vertex x, we list the types of the components of $H - x$. In the left column of the tables, we give the name of the vertex x or, for inner vertices of the gadgets which are not shown in Figure 6, the name of the gadget containing x (S means star-gadget, 2 means 2-gadget, 4 means 4-gadget). In the right column of the tables, S denotes a star of arbitrary size, K_1 an isolated vertex (which is also a star) and K_2 the 1-star, and A/B is either an A or a B. (A) means that A is optional. $A^\cup$ means a (maybe empty) disjoint union of some non-negative number of graphs A. In particular $A^\cup$ might be optional.

Since all these types are contained in some E_j and at most one of the components is neither a star nor a P_4, the table proves that (iii) is true for every digraph obtained from a digraph of type E_i by deleting a vertex.

Observe that every subdigraph of the P_4 or a star is the P_4 or a star. Thus, by induction, (iii) is true for every subdigraph of a digraph of type E_i, and, actually, for every subdigraph of a digraph for which (iii) is true. □

This completes the proof of (iii)$\Longrightarrow$(i), thus the whole proof of Theorem 1 is complete.

4 $[A, -]$-Perfect Forests: Proof of Theorem 2

Similar to the proof technique in Theorem 1, we will prove the four implications (i)$\Longrightarrow$(i')$\Longrightarrow$(ii)$\Longrightarrow$(iii)$\Longrightarrow$(i) of Theorem 2 separately.

In Figure 10 we display the permitted types for the game $[A, -]$. Note that in a *reduced multiple in-star* the vertices v, x and z exist, whereas in a general digraph of type E_1^A, which is a connected induced subdigraph of a reduced multiple in-star, the vertices v, x or z need not exist. In Figure 9 we display the additional forbidden types.

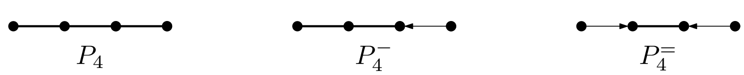

Figure 9: The 3 in-P_4s

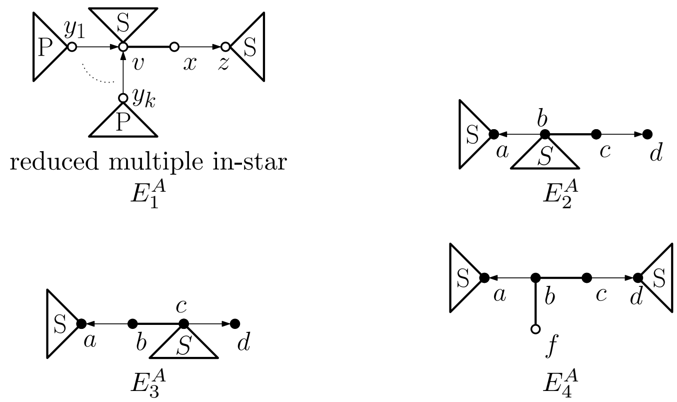

Figure 10: The permitted types. In E_1^A, every vertex could be optional, under the constraint that E_1^A is an induced connected subdigraph of the configuration depicted above.

Proof of Theorem 2(i)⟹(i'). For any digraph D, we know by (2) that every $[A, B]$-perfect digraph is $[A, -]$-perfect. □

Proof of Theorem 2(i')⟹(ii). Let D be $[A, -]$-perfect. By (2), D is also $[A, A]$-perfect. By Theorem 1, D does not contain F_4, $F_{3,1}$, $F_{3,2}$, or $F_+^{(3)}$ as induced subdigraph. It remains to show that Bob has a winning strategy with 2 colours for the game $[A, -]$ on any of the three in-P_4s. This is trivial: Alice is forced to colour a vertex in her first move. Then Bob can colour a vertex at distance 2 with the other colour and wins. □

Proof of Theorem 2(ii)⟹(iii). Note that F_4, $F_{3,1}$, $F_{3,2}$, and $F_+^{(3)}$ are all included in the twenty-four forbidden types of Theorem 1(ii). Moreover, it can be easily checked that any of the remaining seventeen forbidden types given in Theorem 1(ii) contains at least one of the three in-P_4s. Therefore, (ii) implies Theorem 1(ii). Since P_4 is forbidden by (ii), all components of D are undirected stars, except maybe one which is of type E_i for some $i \in \{1, 4, 5, 8, 9\}$ (the other E_is are excluded since they contain an in-P_4).

Since Theorem 1(ii) is true for D, so are Lemma 12 and 13. Moreover, since the in-P_4s are forbidden, we can state stronger versions of these lemmas:

Lemma 34 (Out-Arc-P-Gadget Lemma)**.** *Let (v, w) be a single arc in a tree for which (ii) holds. Assume that the truncated w-branch H_v containing v does not contain any single arc. Then H_v is a P-gadget at v.*

Lemma 35 (In-Arc-Star-Gadget Lemma)**.** *Let (v, w) be a single arc in a tree for which (ii) holds. Assume that the truncated v-branch H_w containing w does not contain any single arc. Then H_w is a star at w.*

By those lemmas, if T of type E_1, then it is of type E_1^A, and if it is of type E_8 or E_9, then it is of type E_4^A.

Finally, if T is of type E_4 (resp. E_5), then observe that the 2-gadget at b (resp. c) cannot be a P_3, since it would form a P_4 with the edge bc. Thus this 2-gadget is a star, which implies that T is of type E_2^A (resp. E_3^A).

Summarizing, T is of type $E_1^A, \dots, E_4^A$. □

Proof of Lemma 34. Since H_v is undirected and P_4 is forbidden by (ii), H_v has diameter at most 2. Thus H_v is a star. □

Proof of Lemma 35. Since H_w is undirected and P_4 is forbidden by (ii), H_w has diameter at most 2. Thus H_w is a star. Since, by (ii), the v-branch containing w does not contain any in-P_4, H_w does not have a P_3 at w. In particular, H_w does not have a pending star at w. □

Proof of Theorem 2(iii)$\Longrightarrow$(i). For the proof it is sufficient to first remark in Observation 36 that the set of permitted digraphs given in (iii) is hereditary and then to show in Propositions 37 – 39 that every digraph of type $E_1^A, \dots, E_4^A$ is $[A, B]$-nice.

Observation 36. *Let H be a digraph of type E_i^A $(1 \le i \le 4)$ and x be one of its vertices. If $H - x$ is non-empty, one of the components of $H - x$ is of type $E_1^A, \dots, E_4^A$, every other component of $H - x$ is a star.*

Type E_1^A	
x	comp. of $H - x$
P	$K_1^\cup, E_1^A$
y_i	$K_2/K_1^\cup, E_1^A$
v	$S^\cup, K_1^\cup, (E_1^A)$
S	E_1^A
x	$E_1^A, (S)$
z	$E_1^A, K_1^\cup$
S	E_1^A

Type E_2^A	
x	comp. of $H - x$
S	E_2^A
a	$K_1^\cup, E_1^A$
b	$S, K_1^\cup, E_1^A$
c	E_1^A, K_1
d	E_1^A
S	E_2^A

Type E_3^A	
x	comp. of $H - x$
S	E_3^A
a	$K_1^\cup, E_1^A$
b	$S, K_1^\cup, E_1^A$
c	$E_1^A, K_1^\cup, K_1$
d	E_1^A
S	E_3^A

Type E_4^A	
x	comp. of $H - x$
S	E_4^A
a	$E_1^A, K_1^\cup$
b	$E_1^A, (K_1), S$
c	S, E_1^A
d	$K_1^\cup, E_1^A$
S	E_4^A
f	E_4^A by def.

Table 2: The proof of Observation 36: Listing the types of the components of $H - x$ for any digraph H from each type E_i^A and any vertex x from H.

Proof. The proof is given in Table 2. The method is similar to the proof of Lemma 16. In Table 2, P denotes a P-gadget. For the other notation we refer to the proof of Lemma 16. □

Proposition 37. *E_1^A is $[A, B]$-nice.*

Proof. Let T be a digraph of type E_1^A. If $\omega(T) = 1$, then T is an in-star, on which Alice wins with one colour if she colours the sink in her first move. Therefore we may assume $\omega(T) = 2$.

We call U the set of unsafe vertices of T that are different from v. Observe that every component of $T - v$ (if v does not exist, $T - v = T$) contains at most one vertex of U, and that this vertex is not an out-neighbour of v. The strategy for Alice is as follows: if v exists, she colours it first; otherwise she colours an arbitrary vertex of U (in the case there is none, Alice wins trivially since all vertices are safe). Then, each time Bob colours a vertex w, Alice colours the vertex of U that is in the same component of $T - v$ as w, if it exists and is uncoloured; otherwise she colours any other uncoloured vertex of U. When every vertex of U is coloured, all uncoloured vertices are safe, therefore Alice wins. □

Proposition 38. *E_2^A and E_3^A are $[A, B]$-nice.*

Proof. Let T be a digraph of type E_2^A or E_3^A. The following is a winning strategy for Alice for the game $[A, B]$ with 2 colours on T. In her first move she colours a. No matter what Bob does, after her second move, Alice can ensure that the center b or c of the star, respectively, is coloured. Then every uncoloured vertex is safe and Alice wins. □

Proposition 39. *E_4^A is $[A, B]$-nice.*

Proof. Let T be a digraph of type E_4^A. A winning strategy of Alice for the game $[A, B]$ with 2 colours on T is the following. In her first move she colours d. Then we have two cases:

- If f exists and Bob colours c or f, then Alice colours the other one with the same colour;
- after that or otherwise, Alice ensures that a is coloured after her next move and that she does not colour c or f herself.

At this point, either all the uncoloured vertices are safe, or b is the only one that is not and either b has no coloured in-neighbour yet or all its in-neighbours are coloured with the same colour. In all cases, Alice ensures that b is coloured after her third move. Then she will win. □

This completes the proof of Theorem 2 (iii)$\Longrightarrow$(i). □

This completes the proof of Theorem 2.

5 Bob begins: Proof of Theorems 41, 42, and 43

Game-perfect forests for the games where Bob begins can be characterised trivially because of the following Observation.

Observation 40 ([6])**.** *A game-perfect digraph D with regard to a game $[B, Y]$ where Bob begins is an undirected graph, i.e. D does not contain any single arc.*

Proof. The digraph consisting of two vertices a, b and a single arc (a, b) is $[B, Y]$-forbidden: it has clique number 1 and a winning strategy for Bob with one colour is to colour a in his first move. □

Observation 40 leads to the following characterisations of game-perfect forests with regard to the games where Bob begins.

Theorem 41. *For a semiorientation D of a forest, the following are equivalent.*

(i) D is $[B, A]$-perfect.

(ii) D does neither contain any single arc nor the chair nor P_5 as an induced subdigraph.

(iii) Every component of D is a P_4 or a star.

Theorem 42. *For a semiorientation D of a forest, the following are equivalent.*

(i) D is $[B, -]$-perfect.

(ii) D does neither contain any single arc nor the chair nor P_5 nor $P_4 \cup K_1$ as an induced subdigraph.

(iii) Either D is the P_4 or every component of D is a star.

Theorem 43. *For a semiorientation D of a forest, the following are equivalent.*

(i) D is $[B, B]$-perfect.

(ii) D does neither contain any single arc nor P_4 as an induced subdigraph.

(iii) Every component of D is a star.

Proof of Theorem 41. **(i)⟹(ii)** Let D be $[B, A]$-perfect. By Observation 40, D does not contain single arcs. By (2), D is $[A, A]$-perfect, thus, by Theorem 1, D does neither contain P_5 nor the chair as an induced subdigraph.

(ii)⟹(iii) Let T be a component of D. Since single arcs are forbidden by (ii), T is an undirected tree. Since P_5 and chair are forbidden by (ii), by Lemma 11 T is a star or a P_4.

(iii)⟹(i) Since every proper subgraph of the P_4 or a star is a forest of stars, (iii) also holds for every subdigraph of D whenever (iii) holds for D. Therefore it is sufficient to describe a winning strategy for Alice with 2 colours on D: using skipping moves Alice can force Bob to start playing on each component of D and then reply in this component to make it safe.

□

Proof of Theorem 42. **(i)⟹(ii)** Let D be $[B,-]$-perfect. By (2), D is $[B,A]$-perfect, thus, by Theorem 41, D does neither contain a single arc nor an induced chair nor an induced P_5. The graph $P_4 \cup K_1$ is also $[B,-]$-forbidden: Bob wins if he colours the isolated vertex in his first move and thus forces Alice to begin colouring the P_4.

(ii)⟹(iii) By Theorem 41, every component of D is a P_4 or a star. If there is a P_4-component, it is the unique component since $P_4 \cup K_1$ is forbidden by (ii).

(iii)⟹(i) By the same argument as in the proof of Theorem 41(iii)⟹(ii) it is sufficient to show that Alice has a winning strategy on D. On the P_4, Alice wins in the game $[B,-]$ with two colours. On a forest of stars, Alice wins obviously.

□

Proof of Theorem 43. **(i)⟹(ii)** Let D be $[B,B]$-perfect. By Observation 40, D does not contain single arcs. By (2), D is $[A,-]$-perfect, thus, by Theorem 2, D does not contain P_4 as an induced subdigraph.

(ii)⟹(iii) Let T be a component of D. Since single arcs are forbidden by (ii), T is an undirected tree. Since P_4 is forbidden by (ii), the diameter of T is at most 2, thus T is a star.

(iii)⟹(i) On a forest of stars, Alice wins obviously.

□

6 Weakly game-perfect forests

Yang and Zhu [22] introduced the following digraph colouring game, which we call *weak digraph colouring game*, whereas the digraph colouring game considered so far is also called *strong digraph colouring game*. Two players, Alice and Bob alternately colour vertices of a given digraph D with colours of a given colour set C, obeying the rule that creating any monochromatic cycle is forbidden. When no more moves are possible, the game ends. Alice wins if every vertex is coloured at the end, otherwise, Bob wins. The smallest cardinality $|C|$ of the colour set such that Alice has a winning strategy is called the *weak game chromatic number* $\chi_{wg}(D)$.

As for the strong game we may also consider six variants wg of the weak digraph colouring game, where $wg = w[X,Y]$ with $g = [X,Y]$ and $X \in \{A,B\}$ and $Y \in \{A,B,-\}$ has the same meaning concerning the player X who begins and the player Y who is allowed to skip as in the strong digraph colouring game.

A notion of game-perfectness for the weak game was introduced in [7]. For any g, a digraph D is *weakly g-perfect* (or *weakly game-perfect with respect to the game g*) if, for any induced subdigraph H of D, $\chi_{wg}(H) = \omega(H)$.

Observation 44. *The inclusions given in (2) for the classes of strongly game-perfect digraphs also hold for the classes of weakly game-perfect digraphs.*

Guo and Surmacs [18] call the weak game chromatic number also *game dichromatic number* because in a natural way it seems to be nearer to the definition of the dichromatic number than the strong game chromatic number. Their definition is justified and supported by the following two results.

Theorem 45 (Yang and Zhu [22])**.** *For (any g and) any orientation D of a graph G,*

$$\chi_{wg}(D) \leq \left\lfloor \frac{\mathrm{col}_g(G)}{2} \right\rfloor,$$

where $\mathrm{col}_g(G)$ denotes the game colouring number introduced by Zhu [23].

Theorem 46 ([7])**.** *For any g, a digraph D is weakly g-perfect if and only if*

(i) the symmetric part $S(D)$ of D is a g-perfect graph and

(ii) D does not contain any induced directed n-cycle with $n \geq 3$.

Since semiorientations of forests do not contain induced directed cycles of length greater than 2, Theorem 46 immediately implies the following.

Corollary 47. *For any g, a semiorientation D of a forest is weakly g-perfect if and only if $S(D)$ is g-perfect.*

Corollary 47 enables us to characterise weakly game-perfect forests. For the proofs of the following characterisations (Theorem 49, 51 resp. 52), recall from the definitions at the beginning that P_4 and stars always denote undirected graphs, whereas a *forest* denotes a digraph (a semiorientation of an undirected forest). In the proofs we frequently use the fact that the strong game and the weak game are equivalent when played on undirected graphs.

Observation 48. *For any undirected graph (=symmetric digraph) G we have $\chi_g(G) = \chi_{wg}(G)$.*

Proof. In both colouring games on a graph G, the vertices of any edge, which is a directed 2-cycle, must be coloured differently. Thus the players have to respect that the colouring is proper, which means that both games are equivalent to Bodlaender's graph colouring game when played on a symmetric digraph. □

Theorem 49. *For a semiorientation D of a forest, the following are equivalent.*

(i) D is weakly $[B, A]$-perfect.

(i') D is weakly $[A, A]$-perfect.

(ii) D does neither contain P_5 nor the chair as an induced subdigraph.

(iii) Every component of $S(D)$ is a star or a P_4.

Proof. The implication (i)$\Longrightarrow$(i') follows directly from Observation 44.

Let D be weakly $[A, A]$-perfect. By Corollary 47, $S(D)$ is $[A, A]$-perfect. By Theorem 1, $S(D)$ does neither contain any induced (undirected) P_5 nor any induced (undirected) chair, which implies (ii). This proves the implication (i')$\Longrightarrow$(ii).

Let D be a digraph that does neither contain P_5 nor the chair as an induced subdigraph. Since D is a forest, every induced P_5 resp. chair in $S(D)$ is an induced subdigraph of D, too. Therefore $S(D)$ does neither contain an induced P_5 nor an induced chair. This means that every component of $S(D)$ has diameter at most 3, and if it has diameter 3, it is a P_4. Thus (ii) implies (iii).

Assume (iii) holds. By Observation 17, $S(D)$ is $[B, A]$-perfect. By Corollary 47, D is weakly $[B, A]$-perfect. Thus (iii) implies (i). $\square$

For the next theorem, we need the following notion. A P_4^0 is a digraph on 5 vertices consisting of an undirected P_4 and an additional vertex v_0 and at most one single arc, which, in case it exists, connects v_0 and some vertex of the P_4. Obviously, there are exactly five pairwise nonisomorphic digraphs that are a P_4^0 (see Figure 11).

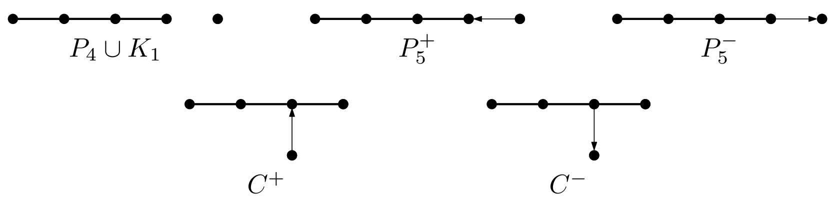

Figure 11: The five digraphs of type P_4^0

Lemma 50. *For any digraph D of type P_4^0 we have $\chi_{w[B,-]} > 2$.*

Proof. For any digraph D of type P_4^0, there exists an edge in D such that its two ending vertices cannot have the same colour. Therefore, Bob must win the weak $[B, -]$-game on D with 1 colour. Now we give a winning strategy for Bob in the weak $[B, -]$-game with 2 colours played on a P_4^0. In his first move, Bob colours v_0. This move does not affect the colouring of any other vertex since v_0 is not contained in a directed cycle. Now Alice is forced to start colouring the P_4, say vertex u. Then Bob wins by colouring a vertex at distance 2 in $S(P_4^0)$ from u. $\square$

Theorem 51. *For a semiorientation D of a forest, the following are equivalent.*

(i) D is weakly $[B, -]$-perfect.

(ii) D does neither contain P_5 nor the chair nor any of the five P_4^0s as an induced subdigraph.

(iii) Either D is the P_4 or every component of $S(D)$ is a star.

Proof. Let D be weakly $[B,-]$-perfect. By Corollary 47, $S(D)$ is $[B,-]$-perfect. By Theorem 42, $S(D)$ does neither contain any induced P_5 nor any induced chair, which implies that D does neither contain an induced P_5 nor an induced chair. By Lemma 50, D does not contain any induced P_4^0. This proves the implication (i)$\Longrightarrow$(ii).

Now, let D be such that (ii) holds. Since D is a forest, every induced P_5 or chair of $S(D)$ is induced in D, too. With (ii) this implies that $S(D)$ does neither contain an induced P_5 nor an induced chair. By Theorem 49, every component of $S(D)$ is a star or a P_4. As well, since D does not contain any induced P_4^0, $S(D)$ does not contain an induced $P_4 \cup K_1$. If $S(D)$ contains a P_4-component H, then, since $P_4 \cup K_1$ is forbidden in $S(D)$, $S(D)$ must be connected and consist only of a P_4. Since D has the same vertex set as $S(D)$, the digraph D is a forest on four vertices that has the P_4 as a subdigraph. Therefore, $D = S(D) = P_4$. Thus (ii) implies (iii).

Finally, let D be such that (iii) holds. Since every proper subdigraph H of D is a digraph with each component of $S(H)$ being a star, we are left to prove that D is $w[B,-]$-nice. If D is the P_4, then Alice wins with 2 colours, since Bob is forced to start colouring the P_4. Otherwise, $S(D)$ is a forest of stars. Then Alice has the following winning strategy. Whenever Bob starts colouring a star of $S(D)$, Alice colours the center of this star if possible. If this is not possible, she colours the center of any other star or any vertex of a star the center has already been coloured. If such move is not possible, every vertex is coloured. Alice will win by following this strategy on $S(D)$. Thus $S(D)$ is $[B,-]$-perfect. Therefore, by Corollary 47, the digraph D is $[B,-]$-perfect. Thus (iii) implies (i). □

Theorem 52. *For a semiorientation D of a forest, the following are equivalent.*

(i) D is weakly $[B,B]$-perfect.

(i') D is weakly $[A,B]$-perfect.

(i") D is weakly $[A,-]$-perfect.

(ii) D does not contain P_4 as an induced subdigraph.

(iii) Every component of $S(D)$ is a star.

Proof. The implications (i)$\Longrightarrow$(i')$\Longrightarrow$(i") follow directly from Observation 44.

Let D be weakly $[A,-]$-perfect. By Corollary 47, $S(D)$ is $[A,-]$-perfect. By Theorem 2, $S(D)$ does not contain an induced P_4, which implies that D does not contain an induced P_4. This proves the implication (i")$\Longrightarrow$(ii).

Now, let D be such that (ii) holds. Since D is a forest, every induced P_4 of $S(D)$ is induced in D, too. With (ii) this implies that $S(D)$ does not contain an induced P_4. Thus every component of $S(D)$ has diameter at most 2, i.e. it is a star. Thus (ii) implies (iii).

Finally, let D be such that (iii) holds. Then, by Theorem 43, $S(D)$ is $[B,B]$-perfect. Thus, by Corollary 47, D is $[B,B]$-perfect. Thus (iii) implies (i). □

7 Infinite game-perfect forests

In this paper, until now, we have characterised the strongly resp. weakly game-perfect semiorientations of finite forests. These results can be easily generalized to semiorientations of infinite forests.

To make clear the notion, an *infinite digraph* (V, A) consists of a vertex set V of arbitrary size (finite or infinite) and an arc set

$$A \subseteq (V \times V) \setminus \{(v, v) \mid v \in V\}.$$

It is called a *proper infinite digraph* if the vertex set V does not have finite cardinality.

When dealing with an infinite digraph D, the rules of the games are modified in the following obvious way. Alice and Bob alternately colour uncoloured vertices of D with a colour from a given, finite colour set C in a feasible way. *Feasible* means in the case of the strong digraph colouring game, the colour of the vertex to be used must be different from the colours of its already coloured in-neighbours, whereas in the case of the weak digraph colouring game, *feasible* means that by colouring a vertex no monochromatic directed (finite) cycle is created. Bob wins the strong game if at some state of the game there is an uncoloured vertex v that has in-neighbours of all colours. Bob wins the weak game if at some state of the game there is an uncoloured vertex v that is contained in nearly monochromatic directed cycles of every colour, where *nearly monochromatic* means that every vertex of the cycle except for v is coloured (by the same colour). For short, in both games, we call such a vertex a *B-win vertex*. Thus, Alice wins if either the digraph is finite and every vertex is coloured finally or the digraph is a proper infinite digraph and the game can be played for an arbitrary number of turns without creating a B-win vertex.

As for the games on finite digraphs we may define the *strong* resp. *weak game chromatic number* of D as the smallest number of colours such that Alice has a winning strategy for the strong resp. weak digraph colouring game or infinity if such number does not exist. The *clique number* of D is the number of vertices in a largest symmetric clique of D or infinity if the sizes of the symmetric cliques in D are not bounded. The infinite digraph D is *strongly* resp. *weakly game-perfect* if, for any induced subdigraph H of D, the strong resp. weak game chromatic number of H equals the clique number of H.

For forests, it is easy to generalize known results from finite digraphs to infinite digraphs.

Theorem 53 ([1])**.** *Every orientation of a possibly infinite forest has strong game chromatic number at most 3.*

Theorem 54 ([1])**.** *Every possibly infinite, undirected forest has game chromatic number at most 4.*

Theorem 54 generalizes a result by Faigle et al. [16] concerning finite forests to the infinite case. In the same way, we can state the following.

Theorem 55. *The characterisations of Theorems 1, 2, 41, 42, 43, 49, 51 and 52 are still valid if D is a semiorientation of an infinite forest.*

Proof. Since in the characterisations of this paper the diameter of the game-perfect forests is bounded, the only case where infinity can come into the forests are the star gadgets. Obviously, in our strategies, whenever stars of arbitrary finite size are allowed, also stars of any infinity cardinality are allowed and do not give additional restrictions to the game. □

8 Final remarks and open questions

A *cactus* is a graph with the property that any two different of its cycles intersect in at most one vertex. In particular, an undirected forest is a cactus without any cycles. Combining the ideas from this paper with the characterisation of strongly game-perfect semiorientations of cycles in [6], it might be possible to easily solve the following problem.

Problem 56. *Characterise game-perfect semiorientations of cactuses for any of the 12 game variants.*

Moreover, the following more general problem, which partially already was proposed in [6], could be the next step towards a characterisation of all game-perfect digraphs for each of the 12 game variants.

Problem 57. *Characterise game-perfect digraphs with clique number 2 for any of the 12 game variants.*

Problem 58. *Characterise game-perfect digraphs for any of the 12 game variants.*

The solution of Problem 58 is known only for the variants $[B, B]$ ([5]) and $[B, -]$ ([9, 19]) of the strong digraph colouring game and for the variants $[B, B]$, $[A, B]$, $[A, -]$ ([5, 7]) and $[B, -]$ ([7, 9, 19]) of the weak digraph colouring game, whereas for the other six, quite more interesting game variants it is still open.

Our results support the following seemingly intuitive conjecture, but which, to our knowledge, still has not been proven.

Conjecture 59. *For any g, if D is strongly g-perfect, then D is weakly g-perfect.*

Or, more generally

Conjecture 60. *For any g, $\chi_{wg}(D) \leq \chi_g(D)$.*

Let $GP[X, Y]$ be the class of $[X, Y]$-perfect digraphs. Our results support the following, which is true for undirected graphs.

Conjecture 61. $GP[A, B] = GP[A, -]$.

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