Variance reduced Value at Risk Monte-Carlo simulations

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Abstract: Monte-Carlo simulations of the risk measure Value at Risk inherently involve standard errors that depend on the sample size N. In this article, we present a variance reduction technique for the estimation of loss probabilities using importance sampling. For a given sample size N, the method reduces the empirical variance of these loss probabilities by more than two orders of magnitude. Thus, it yields more accurate estimators than a standard Monte-Carlo simulation.

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1 Introduction

Value at Risk (VaR) intends to represent the total risk of financial instruments. Monte-Carlo simulations are one method to compute Value at Risk estimators. These estimators naturally involve standard errors. These standard errors typically decrease with an increasing sample size N.

Importance sampling is a variance reduction technique that is aimed at reducing the empirical variance of Monte-Carlo estimators for a given sample size. In this article we will apply importance sampling to the estimation of loss probabilities that can be interpreted as quantiles of loss distributions, i.e. as Value at Risk estimates. Our method reduces the empirical variance of these estimators significantly. Therefore, it yields more accurate estimators with lower standard errors than a standard Monte-Carlo simulation for a given sample size N. The technique employed has been introduced by Singer and has been applied to several other applications already, e.g. to option pricing or the Ginzburg-Landau model of superconductivity [1, 2]. Alternative methods for variance reduced VaR Monte-Carlo simulations already exist [3, 4].

The article is organized as follows: First, we introduce the concept of VaR in section 2. Section 3 introduces the basic concepts of importance sampling, derives a variance reduced VaR Monte-Carlo estimator and introduces suitable elasticity approximations required for the variance reduced Monte-Carlo simulation. In section 4, we present a numerical example that demonstrates the functionality of the presented approach. We discuss results in section 5. The article concludes in section 6.

2 The risk measure VaR

VaR is a risk measure that aims to aggregate the entire risk of a portfolio with value $X(t)$ into one single indicator. It coincides with the loss a portfolio will not exceed at a given confidence level α over a given period of time. Consequently, the VaR corresponds to the (1-α)-quantile of the loss distribution of the portfolio value [5].

Usually, the VaR measure is used to indicate the risk of an entire portfolio. However, in some cases VaR figures are calculated for single financial instruments. For example, regulatory authorities require financial institutions to provide individual VaR estimates for PRIIPs (Packaged Retail and Insurance-based Investment Products)[2]. Going forward, we will focus on portfolios. Of course, this involves the

[2]For details, see [6]. Descriptions of typical PRIIPs can be found at [7, 8].

special case of portfolios that consist of one single financial instrument only.

Several methods exist to provide VaR estimates [5]. The most popular approach is the non-parametric historical simulation. Historical data is employed to estimate the future value development of the considered portfolio. Also parametric approaches exist, e.g., the variance-covariance method. The focus of this article is a third – also parametric – approach, the Monte-Carlo simulation. This approach yields VaR estimators involving standard errors. The size of these standard errors is proportional to $1/\sqrt{N}$ where N is the sample size.

3 Importance sampling of VaR estimators

Importance sampling is a powerful approach to reduce the standard error of Monte-Carlo estimators. In this section, we first briefly introduce the general concept of importance sampling. Subsequently, we derive a variance reduced VaR Monte-Carlo estimator and provide suitable elasticity approximations required for the presented importance sampling approach.

3.1 Basics of importance sampling

The expectation value of a function $h : \mathbb{R}^d \to \mathbb{R}, X \to h(X)$ of a random variable X with probability density p is defined as

$$\alpha = \mathbb{E}_p \left[h\left(X\right)\right] = \int h\left(x\right)\ p\left(x\right)\ \mathrm{d}x. \tag{1}$$

An unbiased Monte-Carlo estimator of this quantity is

$$\hat{\alpha}_p = \frac{1}{n} \sum_{i=1}^{N} h\left(X_i\right) \tag{2}$$

for i.i.d. realizations $X_1, \ldots, X_N$ of X.

Expanding the integrand of equation (1) with any other probability density p' leads to

$$\alpha = \int h\left(x\right)\ p\left(x\right)\ \mathrm{d}x = \int h\left(x\right)\ \frac{p\left(x\right)}{p'\left(x\right)} p'\left(x\right)\ \mathrm{d}x = \mathbb{E}_{p'} \left[h\left(X\right)\ \frac{p\left(X\right)}{p'\left(X\right)}\right]. \tag{3}$$

It can be shown, that the empirical variance of the unbiased Monte-Carlo estimator

$$\hat{\alpha}_{p'} = \frac{1}{n} \sum_{i=1}^{N} h\left(X_i\right)\ \frac{p\left(X_i\right)}{p'\left(X_i\right)} \tag{4}$$

vanishes, if, for a non-negative function h, we choose

$$p'(x) \propto h(x)\, p(x). \tag{5}$$

By normalizing, the product $h(x)\, p(x)$ can be transformed into a probability density [9].

An importance sampling approach introduced by Singer [1, 2] aims to conduct variance reduced Monte-Carlo simulations to calculate integrals of the form

$$P(X(t), t) = \int h\,[X(T)]\, p[X(T), T|X(t), t]\mathrm{d}X(T) \tag{6}$$

where the random process $X(t)$ follows an Itō differential equation of the following type:

$$dX(t) = f[X(t)]\mathrm{d}t + g[X(t)]\mathrm{d}W(t) \tag{7}$$

In this equation, f is the drift vector, t the time, g the diffusion coefficient and $W(t)$ a multivariate Wiener process.

In a lengthy calculation, it can be shown that variance reduction can be achieved by adding an additional drift term to equation (7):

$$dX(t) = (f[X(t)] + \delta f)\,\mathrm{d}t + g[X(t)]\mathrm{d}W(t) \tag{8}$$

For the optimal additional drift term δf, one obtains

$$\delta f = \Omega \nabla \ln P \tag{9}$$

with the diffusion matrix $\Omega = gg^T$. The task of the additional drift term is to "drag" the stochastic process $X(t)$ to areas where the density p' described in equation (5) takes on high values [1, 2].

To conduct a Monte-Carlo simulation with the purpose of estimating the quantity P, a discretization of equation (8) using a Euler-Maruyama approximation [10] is required:

$$X_{k+1} = X_k + (f_k + \delta f_k)\,\Delta\tau + g_k \Delta W_k \tag{10}$$

The following estimator for the integral in equation (6) with sample size N and n discretization steps results:

$$\hat{P} = \frac{1}{N}\sum_{i=1}^{N} \exp\left\{-\sum_{k=0}^{n-1}\left(\frac{1}{2}\delta f_k^T \left|\Omega_k\right|^{-1} \delta f_k \Delta\tau + \delta f_k^T \left|\Omega_k\right|^{-1} g_k \Delta W_k\right)\right\} h[X_{in}] \tag{11}$$

The exponential function corresponds to the term p/p' in equation (4).

A detailed derivation of these formulae can be found in [2] and [11].

3.2 A variance reduced VaR Monte-Carlo estimator

The next step is to derive a variance reduced VaR Monte-Carlo estimator. We assume that the considered portfolio at time t has the value $X(t)$. The total loss L of the portfolio's value in the interval $[0, T]$ is defined as

$$L := -\Delta X = -[X(T) - X(0)]. \tag{12}$$

Our aim is to estimate the probability P, that in this interval the loss L exceeds a given level b, i.e. to estimate $P(L > b)$. Using the indicator function I and the transition density[3] $p[X(T) \mid X(0)]$, this quantity can be expressed as

$$P[L > b \mid X(0)] = \int_0^\infty I(L > b)\, p[X(T) \mid X(0)]\, \mathrm{d}X(T). \tag{13}$$

For simplicity, we assume that the stochastic process $X(t)$ follows a Geometric Brownian motion with drift rate $f = rX$ and diffusion rate $g = \sigma X$. This justifies 0 as the lower integration limit instead of $-\infty$. We also assume that $X(t)$ follows a univariate stochastic process. However, also more complex, multivariate models, e.g., models with stochastic volatility, are compatible with the presented importance sampling approach[4]. Consequently, using equations (9) and (10), we obtain the discretized stochastic differential equation

$$X_{k+1} = X_k + \left(r + \sigma^2 \epsilon_k\right) X_k \Delta\tau + \sigma X_k \Delta W_k \tag{14}$$

with the elasticity

$$\epsilon_k = \frac{X_k}{P(L > b \mid X_k)} \frac{\partial P(L > b \mid X_k)}{\partial X_k}. \tag{15}$$

Using equations (14) and (15) to simulate the portfolio value's trajectories, the following formula results from equation (11) yielding a variance reduced Monte-Carlo estimator of the probability that the total loss L exceeds the given level b:

[3]Note that the transition probability also explicitly depends on the time. Therefore, we should write $p = p[X(T), T \mid X(t), t]$. However, for the purpose of compactness, we suppress the time argument.

[4]For details, see [11, 12].

$$\hat{P}(L > b \mid X_0) = \frac{1}{N}\sum_{i=1}^{N}\exp\left\{-\sum_{k=0}^{n-1}\left(\frac{1}{2}\sigma^2\epsilon_{ik}^2\Delta\tau + \sigma\epsilon_{ik}\Delta W_k\right)\right\} I(L_i > b) \tag{16}$$

3.3 Elasticity approximations for variance reduced VaR estimates

The elasticity ϵ_k is unknown *ex anteriori*, as the probability $P(L > b \mid X_k)$ is generally an unknown quantity. Therefore, we conduct approximations of the elasticity. The quality of the achieved variance reduction highly depends on the specific approximation employed[5]. This article focuses on the approximation by constant values and on the approximation by numerical integration[6].

Approximation by constant values The least complex approach is to approximate the elasticity ϵ by a constant value. Equation (15) indicates that ϵ should take on only negative values as obviously $\frac{\partial P(L>b|X_k)}{\partial X_k} < 0$. (The higher the portfolio value, the lower the probability that the loss will exceed a given threshold.)

Increasing the absolute value of the constant elasticity approximation has two counteracting effects: On the one hand, more trajectories of $X(t)$ will result in a loss L exceeding b and therefore contribute to the Monte-Carlo estimator. On the other hand, the higher the absolute value of ϵ, the higher the discretization bias resulting from the discretized stochastic differential equation (14) will be.

Approximation by numerical integration A more sophisticated approach is the approximation of ϵ by numerical integration. For each trajectory i and each discretization step k we calculate $P(L > b \mid X_k)$ and its partial derivative with respect to X_k by numerical integration employing Gauss-Legendre quadrature.

Based on equation (13) and using discrete time, with definition (12) the loss prob-

[5] An analysis of several possible approximations can be found in [11].

[6] It is important to note, that equation (16) yields an unbiased estimator, independently of the specific choice of ϵ. The choice only influences the estimator's standard error, not its mean value. Of course, for very high absolute ϵ values, a discretization bias is introduced.

ability can be recast as follows:

$$
\begin{aligned}
P(L > b \mid X_k) &= \int_0^\infty I(L > b)\, p[X_n \mid X_k]\, \mathrm{d}X_n \\
&= \int_0^{X_0 - b} p[X_n \mid X_k]\, \mathrm{d}X_n \\
&= 1 - \int_{X_0 - b}^\infty p[X_n \mid X_k]\, \mathrm{d}X_n
\end{aligned} \tag{17}
$$

We assume that $p[X_n \mid X_k]$ is the density of the log-normal distribution [5]. As mentioned above, in this article we only consider Geometric Brownian motions. In this special case, the assumption is accurate. However, also for more complex models of the stochastic process $X(t)$ for the purpose of calculating ϵ this approach is permissible (compare footnote 6). Thus, we obtain

$$
\begin{aligned}
p[X_n \mid X_k] = {} & \frac{1}{\sqrt{2\pi\sigma^2 (n-k)\Delta t}} \frac{1}{X_n} \\
& \times \exp\left\{ - \frac{\left[\ln X_n - \ln X_k - \left(r - \frac{\sigma^2}{2} \right)(n-k)\Delta t \right]^2}{2\sigma^2 (n-k)\Delta t} \right\}.
\end{aligned} \tag{18}
$$

The affine transformation [13]

$$
\begin{aligned}
& X_n = X_0 - b + \frac{1+y}{1-y} \\
& \qquad \Rightarrow \int_{X_0 - b}^\infty f(X_n) \mathrm{d}X_n = \int_{-1}^1 \frac{2}{(1-y)^2} f\left(X_0 - b + \frac{1+y}{1-y} \right) \mathrm{d}y
\end{aligned} \tag{19}
$$

yields

$$
\begin{aligned}
P(L > b \mid X_k) = {} & 1 - \frac{1}{\sqrt{2\pi\sigma^2\,(n-k)\,\Delta t}} \int_{-1}^1 \frac{2}{(1-y)^2} \frac{1}{X_0 - b + \frac{1+y}{1-y}} \\
& \times \exp\left\{ - \frac{\left(\ln\left(X_0 - b + \frac{1+y}{1-y} \right) - \ln X_k - \left(r - \frac{\sigma^2}{2} \right)(n-k)\Delta t \right)^2}{2\sigma^2 (n-k)\Delta t} \right\} \mathrm{d}y.
\end{aligned} \tag{20}
$$

For the purpose of compactness, we abbreviate

$$
\xi := \ln\left(X_0 - b + \frac{1+y}{1-y} \right) - \ln X_k - \left(r - \frac{\sigma^2}{2} \right)(n-k)\Delta t \tag{21}
$$

and obtain

$$
\begin{aligned}
P(L > b \mid X_k) = 1 - \frac{1}{\sqrt{2\pi\sigma^2\,(n-k)\,\Delta t}} \int_{-1}^{1} \frac{2}{(1-y)^2} \frac{1}{X_0 - b + \frac{1+y}{1-y}} \\
\times \exp\left\{-\frac{\xi^2}{2\sigma^2(n-k)\Delta t}\right\} \mathrm{d}y.
\end{aligned}
\tag{22}
$$

This term can be evaluated by means of Gauss-Legendre quadrature [14, 15]. Abbreviating

$$
\xi_i := \ln\left(X_0 - b + \frac{1+x_i}{1-x_i}\right) - \ln X_k - \left(r - \frac{\sigma^2}{2}\right)(n-k)\Delta t \tag{23}
$$

with n_{GL} abscissae x_i and weights w_i, we obtain

$$
\begin{aligned}
P(L > b \mid X_k) \approx 1 - \frac{1}{\sqrt{2\pi\sigma^2\,(n-k)\,\Delta t}} \sum_{i=1}^{n_{\mathrm{GL}}} \frac{2}{(1-x_i)^2} \frac{1}{X_0 - b + \frac{1+x_i}{1-x_i}} \\
\times \exp\left\{-\frac{\xi_i^2}{2\sigma^2(n-k)\Delta t}\right\} w_i.
\end{aligned}
\tag{24}
$$

Analogously, for the partial derivative, by applying Leibniz' rule for parameter integrals [16] the following expression results:

$$
\begin{aligned}
\frac{\partial P(L > b \mid X_k)}{\partial X_k} &= -\frac{1}{\sqrt{2\pi\sigma^2\,(n-k)\,\Delta t}} \int_{-1}^{1} \frac{2}{(1-y)^2} \frac{1}{X_0 - b + \frac{1+y}{1-y}} \\
&\quad \times \frac{\partial}{\partial X_k} \exp\left\{-\frac{\xi^2}{2\sigma^2(n-k)\Delta t}\right\} \mathrm{d}y \\
&= -\frac{1}{\sqrt{2\pi\sigma^2\,(n-k)\,\Delta t}} \int_{-1}^{1} \frac{2}{(1-y)^2} \frac{1}{X_0 - b + \frac{1+y}{1-y}} \\
&\quad \times \exp\left\{-\frac{\xi^2}{2\sigma^2(n-k)\Delta t}\right\} \frac{\xi}{\sigma^2(n-k)\Delta t} \frac{1}{X_k} \mathrm{d}y
\end{aligned}
\tag{25}
$$

Note that for $k = 0$ additional terms appear, as

$$
\frac{\partial \ln\left(X_0 - b + \frac{1+y}{1-y}\right)}{\partial X_0} \quad \text{and} \quad \frac{\partial \left(X_0 - b + \frac{1+y}{1-y}\right)^{-1}}{\partial X_0}
$$

do not vanish. For reasons of simplicity, we suppress these terms and use the same expression for the first discretization step as for all the other discretization steps.

Again, we conduct a Gauss-Legendre quadrature and obtain

$$\begin{aligned}\frac{\partial P(L > b \mid X_k)}{\partial X_k} \approx -&\frac{1}{\sqrt{2\pi\sigma^2\,(n-k)\,\Delta t}} \sum_{i=1}^{n_{\mathrm{GL}}} \frac{2}{(1-x_i)^2} \frac{1}{X_0 - b + \frac{1+x_i}{1-x_i}} \\ &\times \exp\left\{-\frac{\xi_i^2}{2\sigma^2(n-k)\Delta t}\right\} \frac{\xi_i}{\sigma^2(n-k)\Delta t} \frac{1}{X_k} w_i.\end{aligned} \tag{26}$$

With these quadrature results we can calculate ϵ_{ik} for each trajectory i and each discretization step k from equation (15).

Before presenting numerical results, two comments should be made:

- The calculation of $N \times n$ different ϵ_{ik} values involving the Gauss-Legendre quadrature of at least two integrals each is very demanding with regards to computational power. An approach that can significantly reduce the required amount of arithmetical operations is to first calculate an ϵ-grid as a function of the current portfolio value X_k and the remaining discretized time $(n-k)\Delta t$. The required ϵ_{ik} for each trajectory i and each discretization step k can then be obtained by linear intrapolation or by cubic spline intrapolation from this grid.

- In the Black-Scholes model involving a Geometric Brownian motion for the portfolio value $X(t)$, Monte-Carlo simulations are of course not required to calculate quantiles of the log-normal distribution. However, the importance sampling approach presented in this article also works well for more complex models, e.g., models with stochastic volatility. In this case, for the calculation of ϵ, the assumption of a log-normal transition density is a well-working approximation[7].

4 Numerical results

We now present a numerical example that demonstrates the ability of the introduced approach to significantly reduce the empirical variance of VaR Monte-Carlo estimators. We consider the following setting:

- The considered portfolio $X(t)$ has an initial value of $X(0) = 50$.

- The portfolio's value follows a Geometric Brownian motion with interest rate $r = 0.05$ and volatility $\sigma = 0.2$.

[7]Compare [11], section 3.7 for details.

$b = 10$	Benchmark	Constant approximation				Integration
		$\epsilon = -5$	$\epsilon = -7.5$	$\epsilon = -15$	$\epsilon = -20$	
$\hat{P}(L > b)$	9.30%	10.36%	10.79%	10.77%	13.02%	10.33%
$\hat{\sigma}_{\hat{P}}$	0.92%	0.45%	0.42%	0.85%	3.35%	0.13%
$\hat{\sigma}^2_{\hat{P}}$	8.5E-05	2.1E-05	1.7E-05	7.3E-05	1.1E-03	1.7E-06
VarRatio	1.0	4.1	4.8	1.2	0.1	49.5

Table 1: Variance reduced Monte-Carlo simulation of the probability that the portfolio loss exceeds $b = 10$.

$b = 20$	Benchmark	Constant approximation				Integration
		$\epsilon = -5$	$\epsilon = -7.5$	$\epsilon = -15$	$\epsilon = -20$	
$\hat{P}(L > b)$	0.30%	0.37%	0.37%	0.31%	0.38%	0.34%
$\hat{\sigma}_{\hat{P}}$	0.17%	0.05%	0.03%	0.02%	0.03%	0.01%
$\hat{\sigma}^2_{\hat{P}}$	3.0E-06	2.9E-07	1.2E-07	2.9E-08	8.9E-08	4.0E-09
VarRatio	1.0	10.5	24.9	102.4	33.5	740.7

Table 2: Variance reduced Monte-Carlo simulation of the probability that the portfolio loss exceeds $b = 20$.

- The aim is to estimate the probability that the portfolio loss as defined in equation (12) exceeds a given level b for $T = 1$. First we consider a moderate loss ($b = 10$) in table 1 and then a high loss ($b = 20$) in table 2.
- The Monte-Carlo simulation involves $N = 1{,}000$ trajectories and $n = 100$ discretization steps per trajectory.
- For the purpose of Gauss-Legendre quadrature, we choose $n_{\text{GL}} = 64$ abscissae and weights[8].
- For each example (moderate and high loss), we first conduct a benchmark simulation without variance reduction. Then, we conduct a variance reduced simulation employing several constant approximations of the elasticity ϵ. Finally, the approximation of ϵ by Gauss-Legendre quadrature (integration) is considered.
- As results, we present the estimated probability $\hat{P}(L > b \mid X_0)$, its standard error $\hat{\sigma}_{\hat{P}}$, the empirical variance $\hat{\sigma}^2_{\hat{P}}$ and the variance ratio ("VarRatio"), i.e. the ratio between the benchmark's empirical variance and the empirical variance of the approach considered.

[8] Abscissae and weight values are available on [15].

5 Discussion of results

The results indicate that the presented approach is mainly suitable for settings in which the aim is to estimate probabilities of high losses. In a standard Monte-Carlo simulation, only few trajectories yield very high losses. Thus, for the simulation of probabilities of high losses, only few trajectories contribute to the Monte-Carlo estimator but increase its empirical variance. The presented importance sampling approach achieves to drag an increased amount of trajectories into the area of interest, i.e. the area of high losses. This result is in accordance with the finding that the same importance sampling approach in the Monte-Carlo simulation of options delivers the best results for deep-out-of-the-money options [11].

The integration method yields better results than the constant approximation approach. Also this finding is intuitive: the integration method accurately resembles the shape of the ϵ-surface. Contrastingly, the constant approach delivers only a very crude approximation of the true elasticity. Also this result is in line with the variance reduced Monte-Carlo simulation of options [11].

Finally, the optimal constant ϵ value depends on the specific parameters of a Monte-Carlo simulation. While for the moderate loss environment ($b = 10$) a value of $\epsilon = -7.5$ delivers best results, for the high loss environment ($b = 20$) a value of $\epsilon = -15$ performs best. For too high ϵ values the variance is increased compared to the benchmark.

6 Conclusion

The purpose of this article is to introduce an approach that allows to significantly reduce the empirical variance of VaR Monte-Carlo estimators. After laying the theoretical foundation, we derive variance reduced VaR estimators and introduce suitable elasticity approximations as requirement for importance sampling. Numerical results show that this method actually serves to reduce standard errors of estimators, mainly in situations where probabilities of high losses are estimated. The computationally more demanding integration method delivers better results, but also the comparably simple constant approximation method strongly decreases variance.

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